

Asymptotic mixing time analysis of a random walk on the orthogonal group

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November 12, 2012

Abstract

We consider an analogue of the Kac random walk on the special orthogonal group, in which at each step a random rotation is performed in a randomly chosen 2-plane of \mathbb{R}^N . We obtain sharp asymptotics for the rates of convergence in total variance distance, establishing a cutoff phenomenon; in the special case where the rotation angle is fixed this confirms a conjecture of Rosenthal [17]. Under mild conditions we also establish a cutoff for convergence of the walk to stationarity under the L^2 norm. Depending on the distribution of the randomly chosen angle of rotation, several surprising features emerge. For instance, it is sometimes the case that the mixing times differ in the total variation and L^2 norms. Our analysis hinges on a new method of estimating the characters of the orthogonal group, using a contour formula and saddle point estimates.

1 Introduction

The asymptotic analysis of the mixing time of Markov chains is an emerging field, and while discrete space Markov chains have a healthy and growing literature, there are still comparatively few examples of continuous state chains where the mixing time has been completely determined. Among the most natural and first studied such chains is the random walk on the orthogonal group introduced by Mark Kac [12] in order to model the velocities of a large number N of particles making elastic collisions. Briefly, at each step of the walk two velocities v_i and v_j are chosen at random and updated in such a way that $v_i'^2 + v_j'^2 = v_i^2 + v_j^2$. What is the same, the vector of velocities (v_1, \dots, v_N) is multiplied by a matrix from $SO(N)$ which consists of a rotation by a randomly chosen angle, in the randomly chosen coordinate 2-plane, $e_i \wedge e_j$.

The asymptotic mixing time of Kac's walk has received quite a bit of attention, going back at least to the paper of Diaconis and Saloff-Coste [4], but remains only incompletely understood. Besides the evident difficulty that the steps do not commute, the walk poses a number of challenges, foremost among them that, while the spectral gap has been determined by a number of authors [14], [10], [1], the bulk of the spectrum of the transition kernel is not known. Even granting the spectrum, there is the further difficulty that the walk's density is not in L^2 after any finite number of steps, which complicates the use of standard spectral techniques. In fact, only very recently the

second author [11] has given the first polynomial bound for the mixing time in the most natural total variation metric, but at $O(N^5(\log N)^2)$ it is still far from the expected true bound $O(N^2)$.

The purpose of this article is to study a close relative of the Kac walk, which poses some of the same challenges, but for which we can give a sharp asymptotic analysis of the mixing time. In the ‘uniform plane Kac walk’ at each step we perform a rotation by a random angle θ , but now with the plane of rotation chosen uniformly from $\Lambda^2(S^{N-1})$. Formally, let ξ be a Borel probability measure on \mathbb{T}^1 and let

$$R(1, 2; \theta) := \begin{pmatrix} \cos \theta & -\sin \theta & & \\ \sin \theta & \cos \theta & & \\ & & & \\ & & & I_{N-2} \end{pmatrix};$$

the transition kernel of our walk is the map $P_\xi : L^2(SO(N)) \rightarrow L^2(SO(N))$,

$$P_\xi f(X) = \int_{\mathbb{T}^1} \int_{SO(N)} f(UR(1, 2; \theta)U^{-1}X) d\nu(U) d\xi(\theta)$$

where $d\nu$ denotes Haar measure on $SO(N)$.

Theorem 1.1. *Let ξ be any Borel probability measure on \mathbb{T}^1 excluding the delta measure at 0. Define $\sigma(\theta) = \sin(\theta/2)$ and*

$$\xi(\sigma^2) = \int_{\mathbb{T}^1} \sigma^2(\theta) d\xi(\theta).$$

The random walk P_ξ with initial state at the identity on $SO(N)$ has total variation cutoff at $t = N \log N / 4\xi(\sigma^2)$. Precisely, uniformly in N , for $t = N(\log N + c) / 4\xi(\sigma^2)$ we have

$$\|\delta_{Id} P_\xi^t - \nu\|_{\text{TV}(SO(N))} = I(c < 0) + f(c),$$

where $f(c)$ is a function tending to 0 as $|c| \rightarrow \infty$.

The chief simplifying feature of our walk is that the transition kernel is invariant under conjugation, so that the spectrum is completely described by the character theory of $SO(N)$, an observation that goes back to Diaconis and Shahshahani in [5]. Our analysis builds on that of Rosenthal [17], who studied the walk in the special case where ξ is a point mass at a fixed angle θ , proving the lower bound of our theorem in this case, and the more difficult upper bound for $\theta = \pi$ (see also work of Porod on a related walk [16]). The first stage in our proof of Theorem 1.1 is to complete Rosenthal’s upper bound analysis for any fixed angle, with strong uniformity in the angle θ .

Theorem 1.2. *Let $\theta = \theta(N)$ vary with N in such a way that $\frac{\log N}{\sqrt{N}} \leq \theta \leq \pi$. Let P_θ denote the transition kernel of our walk in the special case when ξ is a point mass at θ . For $t = N(\log N + c) / 4\sigma^2(\theta)$, uniformly in N and $\theta(N)$,*

$$\|\delta_{Id} P_\theta^t - \nu\|_{\text{TV}} = I(c < 0) + f(c)$$

where again $f(c) \rightarrow 0$ and $|c| \rightarrow \infty$.

As in the special case $\theta = \pi$ treated by Rosenthal, the upper bound of Theorem 1.2 is obtained by bounding the total variation norm with the L^2 norm, which translates the problem into bounding a sum of character ratios (see [3] for similar computations in the finite group setting). Although this analysis does not go through for the upper bound in Theorem 1.1 because the density need not be in L^2 , we are able to sidestep this issue with a truncation argument (the extra symmetry of the walk is exploited a second time).

The main new technique in our paper is an integral and corresponding differential method for evaluating the character ratios of the orthogonal group at a rotation, that may be of independent interest. Let $\rho_{\mathbf{a}}$ be an irreducible representation of $SO(N)$ indexed by dominant weight \mathbf{a} , of dimension $d_{\mathbf{a}}$ and character $\chi_{\mathbf{a}}$. Rosenthal derived from the Weyl character formula an expression for the ratios

$$r_{\mathbf{a}}(\theta) = \frac{\chi_{\mathbf{a}}(R(\theta))}{d_{\mathbf{a}}}$$

as a trigonometric polynomial, which he could bound successfully when $\theta = \pi$. We observe that in fact the character ratio is equal to the sum of the residues of a meromorphic function attached to the representation, which allows us to use a contour integral and the method of stationary phase to give powerful estimates for the character (see Lemma 2.1 of Section 2.1, see also Lemma 5.1 of Section 5.1 for the differential formulation). To put in perspective the type of estimates that we are able to so obtain, for $\theta \neq \pi$ we show bounds for Rosenthal's trigonometric polynomial which are *exponentially* small in the length of the sum. It is a surprising feature of method that the special case $\theta = \pi$ that was resolved by Rosenthal using combinatorial arguments is most difficult for us to handle using our integral formula!

When the measure ξ of Theorem 1.1 has support bounded away from zero, the resulting walk converges in L^2 and, under the additional assumption that the support of ξ is bounded away from π , we are also able to determine the L^2 mixing time, and establish a cutoff phenomenon.

Theorem 1.3. *Let ξ be a probability measure on \mathbb{T}^1 having support bounded away from 0 and π , and let q be the point in the support of ξ that is closest to 0. Let, as before, $\sigma(\theta) := \sin(\theta/2)$ and $\xi(\sigma^2) = \int_0^{2\pi} \sigma^2(\theta) \xi(d\theta)$. Uniformly in N , we have*

$$t > \max \left(\frac{N \log N}{4\xi(\sigma^2)}, \frac{N(\log N + 2 \log \log N)}{8\sigma^2(q)} \right) + cN \quad \Rightarrow \quad \|\delta_{Id} P^t - \nu\|_2 \leq 1/f(c)$$

and

$$t < \max \left(\frac{N \log N}{4\xi(\sigma^2)}, \frac{N(\log N - 3 \log \log N)}{8\sigma^2(q)} \right) - cN \quad \Rightarrow \quad \|\delta_{Id} P^t - \nu\|_2 \geq f(c)$$

where $f(c)$ is a function tending to ∞ as c does.

Remark 1. When $\sigma^2(q) < \frac{1}{2}\xi(\sigma^2)$ we exhibit a cutoff window of size $O(N \log \log N)$. It is likely that $O(N)$ is the true size.

Together, Theorems 1.1 and 1.3 have several surprising features. For one, the proof of the Theorem 1.3 exhibits a competition between the natural representation, which produces the spectral gap, and some moderately sized representations for which

the spectrum is smaller, but of higher multiplicity. We are not familiar with another such naturally occurring walk that has been studied. Also, for measures ξ such that $\sigma^2(q) < \frac{\xi(\sigma^2)}{2}$ we have a situation in which there is a cutoff phenomenon in both the total variation and L^2 norms, the two cutoff points are not the same, and both may be precisely analyzed. Again, we are not aware of comparable examples.

The proofs of both the lower and upper bound of Theorem 1.3 are strongly dependent upon our integral formula for the character ratio. We prove the lower bound by using the saddle point method to obtain an asymptotic formula for the character ratio of a special class of moderate representations, for which we can also accurately estimate the dimension. The proof of the upper bound shows that this class of special representations is ‘nearly optimal’ with respect to trading off size of character ratio with dimension, by approximating the relative character ratios with the ratios of the respective integrands at the saddle point.

We record several further consequences of Theorems 1.1 and 1.3. First, from Theorem 1.1 one can also derive near optimal mixing time results under the Wasserstein distance. Recall that for two probability measures μ, ν defined on a metric space (Ω, d) , the L^2 Wasserstein distance is defined by

$$W^2(\mu, \nu) = \inf_{(X,Y) \in \mathcal{M}(\mu,\nu)} [\mathbb{E}d(X,Y)^2]^{1/2}$$

where $\mathcal{M}(\mu, \nu)$ denotes the set of all couplings of μ and ν . We obtain

Corollary 1.1. *Let ξ denote the uniform measure on \mathbb{T}^1 . For $t = N(\log N + c)$, the Wasserstein distance of the uniform plane Kac walk $\delta_{Id}P_\xi^t$ from Haar measure is bounded by*

$$W^2(\delta_{Id}P_\xi^t, \nu) = o(1), \quad c \rightarrow \infty.$$

Proof sketch. The optimal transport inequality [19] gives

$$W^2(\mu, \nu) \ll \sqrt{d^2 \|\mu - \nu\|_{TV}}$$

where d is the diameter of the state space. For our walk on $SO(N)$ the diameter is of order \sqrt{N} . It follows in a straightforward way from our analysis that $\|\delta_{Id}P^{N(\log N + c)} - \nu\|_{TV} = o(1/N)$ as $c \rightarrow \infty$, since the second largest eigenvalue is of size $1 - \frac{2}{N} + O(N^{-2})$. \square

Remark 2. Oliviera [15] proved that the Kac walk on $SO(N)$ converges in at most $t = \frac{1}{4}N^2 \log N$ steps under L^2 Wasserstein distance. A direct adaptation of his method would give the same upper bound for the uniform plane Kac walk defined above. Our result does better than this by an order of N .

In a familiar way, the analysis in Theorem 1.3 implies a cut-off phenomenon at twice the L^2 mixing time in the ‘ L^∞ norm with respect to Haar measure’. Note that in the continuous state space setting there is some subtlety in working with L^∞ norms. This is explained in Section 7.

Corollary 1.2. *The conclusions of Theorem 1.3 are valid with the mixing time doubled and the L^2 norm replaced by the L^∞ norm with respect to Haar measure.*

We close the introduction with a brief description of the results in our appendices. While for this investigation we needed to understand the characters of $SO(N)$ evaluated only at a single rotation, in Appendix A we prove a corresponding multiple integral formula for the characters of $SO(N)$ evaluated at an arbitrary conjugacy class. Thus, in principle, the techniques developed here could be used to study a random walk generated by elements of $SO(k) \subset SO(N)$ instead of rotations in a single plane. Suitably modified, the argument of Appendix A also carries over to give integral formulae for the character ratios on the other classical compact Lie groups.

Frobenius gave a well-known formula for the character ratios of the symmetric group S_n evaluated at a cycle, as the residue at infinity of a certain rational function ([13], p. 118). In Appendix B we prove a multiple contour formula for the character ratios of S_n evaluated at an arbitrary conjugacy class, which generalizes Frobenius' result, and is the natural analogue of our formula from Appendix A. This contour has recently been exploited by the first author to bound the mixing time of the k -cycle random walk on S_n [9].

List of Notations

Here we collect many symbols used extensively throughout the article. They will be reintroduced in the subsequent sections.

- θ = the angle of rotation defining the generalized Rosenthal walk.
- $\sigma = \sin(\theta/2)$; the mixing time is proportional to $(2\sigma^2)^{-1}$.
- μ is a probability measure on \mathbb{T}^1 , used to define the mixed Rosenthal walk.
- $N = 2n + 1$ is the dimension of the ambient space of the special orthogonal group $SO(N)$.
- $P = P_\theta$ or P_μ is the corresponding Markov kernel.
- $\mathbf{a} \in \mathbb{N}^n$ is the index of an irreducible representation of $SO(N)$.
- $\mathbf{s} \in \mathbb{N}^n$ is the index of an irreducible representation in shift notation.
- $\tilde{a}_j = a_j + j - 1/2$ is the component of the Rosenthal index for irreps.
- $\alpha_j = \tilde{a}_j/n$ denotes the rescale index component.
- $d_{\mathbf{a}}$ is the dimension of the irreducible representation $\rho_{\mathbf{a}}$.
- $r_{\mathbf{a}}(\theta) = \text{Tr} \rho_{\mathbf{a}}(R(1, 2; \theta)) / d_{\mathbf{a}}$ is the character ratio at \mathbf{a} .
- $g_{\mathbf{a}}(z) = \theta z - \frac{1}{n} \sum_{j=1}^n \log(z^2 + \alpha_j^2)$ is the logarithm of the contour integrand for $r_{\mathbf{a}}(\theta)$.
- $\omega \sim \cot(\theta/2)$ is the critical saddle point of g_0 on the positive real axis.

2 Background and representation theory

For simplicity we restrict to the case $N = 2n + 1$ is odd. All of our arguments go through for even N with simple modifications.

Let μ and μ' be Borel probability measures on $SO(2n + 1)$. Their total variation distance is defined by

$$\|\mu - \mu'\|_{\text{TV}} = \sup_A |\mu(A) - \mu'(A)|,$$

the supremum taken over all Borel sets A . We will be primarily concerned with this metric, but in the case that $|\mu - \mu'|$ has density $\frac{d|\mu - \mu'|}{d\nu}$ with respect to Haar measure, we also define the L^2 distance

$$\|\mu - \mu'\|_{L^2} = \left(\int_{SO(2n+1)} \left(\frac{d|\mu - \mu'|}{d\nu} \right)^2 d\nu \right)^{1/2}.$$

Our results make use of the harmonic analysis on $SO(N)$; a reference for the theory of harmonic analysis on compact groups is [8], see also [18] and [6].

Probability measure μ has a Fourier development in terms of the finite dimensional irreducible representations of $SO(N)$. These are indexed by weakly increasing integer ‘highest weights’

$$\mathbf{a} = (a_1, \dots, a_n), \quad 0 \leq a_1 \leq a_2 \leq \dots \leq a_n.$$

We will also frequently use strongly increasing half-integer weights (Rosenthal’s convention, convenient for the Weyl character formula)

$$\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_n), \quad \tilde{a}_k = a_k + k - 1/2$$

and the shift-indices

$$\mathbf{s} = (s_1, \dots, s_n), \quad s_1 = a_1, \quad s_i = a_i - a_{i-1}, \quad 1 < i \leq n.$$

The representation corresponding to $\mathbf{a} = \mathbf{0}$ is the trivial representation while \mathbf{a} with $a_n = 1$, $a_k = 0$, $k < n$ indicates the natural (matrix) representation. We set $d_{\mathbf{a}}$ for the dimension of irreducible representation $\rho_{\mathbf{a}}$, and let $\chi_{\mathbf{a}}(g) = \text{Tr}(\rho_{\mathbf{a}}(g))$ be the character.

We consider measures μ that are invariant under conjugation. Such measures have a Fourier series in terms of the irreducible characters on $SO(N)$:¹

$$\mu \sim \sum_{\mathbf{a}} \hat{\mu}(\chi_{\mathbf{a}}) \chi_{\mathbf{a}}, \quad \hat{\mu}(\chi_{\mathbf{a}}) = \frac{1}{d_{\mathbf{a}}} \int_{SO(2n+1)} \chi_{\mathbf{a}}(g) \mu(dg).$$

The Fourier map carries convolution to point wise multiplication:

$$\widehat{\mu * \mu'}(\chi_{\mathbf{a}}) = \hat{\mu}(\chi_{\mathbf{a}}) \cdot \hat{\mu'}(\chi_{\mathbf{a}}).$$

Plancherel’s identity takes the form

$$\|\mu - \mu'\|_{L^2}^2 = \sum_{\mathbf{a}} d_{\mathbf{a}}^2 |\hat{\mu}(\chi_{\mathbf{a}}) - \hat{\mu'}(\chi_{\mathbf{a}})|^2,$$

with the interpretation that the right side is finite if and only if $|\mu - \mu'|$ has an L^2 density with respect to Haar measure. The Fourier coefficients of Haar measure are given by $\hat{\nu}(\chi_{\mathbf{0}}) = 1$, and $\hat{\nu}(\chi_{\mathbf{a}}) = 0$ if $\mathbf{a} \neq \mathbf{0}$.

In the special case that $\mu_{\theta} = \delta_{Id} \cdot P_{\theta}$ is the probability measure generating the fixed- θ Rosenthal walk, set $r_{\mathbf{a}}(\theta) = \hat{\mu}_{\theta}(\chi_{\mathbf{a}}) = \frac{\chi_{\mathbf{a}}(R_{\theta})}{d_{\mathbf{a}}}$ for the character ratio at rotation R_{θ} . The starting point for our analysis is the Upper Bound Lemma as discussed in [17].

¹The characters of $SO(N)$ are real. Note that our normalization for $\hat{\mu}(\chi_{\mathbf{a}})$ differs from Rosenthal’s in the factor of $\frac{1}{d_{\mathbf{a}}}$.

Lemma (Upper Bound Lemma). *Let μ be a conjugation invariant probability measure on $SO(2n+1)$ and let ν be Haar measure. Then the total variation distance between μ and ν is bounded by*

$$\|\mu - \nu\|_{\text{TV}} \leq \frac{1}{2} \|\mu - \nu\|_{L^2} = \frac{1}{2} \left(\sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 |\hat{\mu}(\chi_{\mathbf{a}})|^2 \right)^{1/2}.$$

In particular, the total variation distance to Haar measure for the Rosenthal walk of fixed angle θ and step t is bounded by

$$\|\delta_{Id} P_{\theta}^t - \nu\|_{\text{TV}} \leq \frac{1}{2} \left(\sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 r_{\mathbf{a}}(\theta)^{2t} \right)^{1/2}. \quad (1)$$

2.1 Dimension and character ratio formulae

Using L'Hospital's rule and the Weyl character formula, Rosenthal [17] gives the following evaluations of $d_{\mathbf{a}}$ and $r_{\mathbf{a}}(\theta)$:

$$d_{\mathbf{a}} = \frac{2^n}{1!3!\dots(2n-1)!} \prod_{q=1}^n \tilde{a}_q \prod_{1 \leq s < r \leq n} [\tilde{a}_r^2 - \tilde{a}_s^2] \quad (2)$$

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2 \sin(\theta/2))^{2n-1}} \sum_{j=1}^n \frac{\sin(\tilde{a}_j \theta)}{\tilde{a}_j \prod_{r \neq j} (\tilde{a}_r^2 - \tilde{a}_j^2)}. \quad (3)$$

In the special case $\theta = \pi$, he obtains good bounds for $|r_{\mathbf{a}}(\pi)|$ by taking absolute values in the above sum; in the general case, for $\theta \neq 0, \pi$ this no longer gives non-trivial results, since the factor $(\sin(\theta/2))^{2n-1}$ in the denominator grows exponentially with n . In general we are able to estimate the necessary massive cancellation on the sign in light of the observation that $r_{\mathbf{a}}(\theta)$ is exactly the sum of the residues of the meromorphic function

$$f_{\mathbf{a},\theta}(z) = \frac{(2n-1)!}{(2 \sin \theta/2)^{2n-1}} \frac{\sin(\theta z)}{\prod_{j=1}^n [\tilde{a}_j^2 - z^2]}.$$

This fact leads to the following integral formula.

Lemma 2.1. *For any $\alpha > 0$,*

$$r_{\mathbf{a}}(\theta) = \frac{1}{2\pi i} \int_{\Re(z)=\alpha} \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{e^{n\theta z}}{\prod_{j=1}^n (\alpha_j^2 + z^2)} dz \quad (4)$$

where $\alpha_j := \tilde{a}_j/n$.

Proof. We may express the sum of the residues of $f_{\mathbf{a},\theta}$ as

$$r_{\mathbf{a}}(\theta) = \frac{1}{2\pi i} \int_{\mathcal{R}} f_{\mathbf{a},\theta}(z) dz$$

where \mathcal{R} is any rectangle with corners $\pm B \pm in\alpha$, $B > \tilde{a}_n$, oriented counter-clockwise. As $B \rightarrow \infty$, the integral over the vertical segments go to 0 leaving two horizontal line integrals at $\Im(z) = \pm in\alpha$. Exchanging $z \rightarrow -z$, the two integrals are seen to be equal, so we keep twice the top one. Now split $\sin(\theta z)$ as $\frac{e^{i\theta z} - e^{-i\theta z}}{2i}$; the contribution from $e^{i\theta z}$ vanishes by shifting the contour upward to $i\infty$. For the final result, replace z by iz/n . \square

3 Sketch of proof: the trivial character

Recall that for the fixed θ walk we set $\sigma = \sin(\theta/2)$, and that for Theorem 1.2 we need to prove, uniformly in n , that for $t = n(\log n + c)$, $\|\delta_{\text{Id}} P_\theta^t - \nu\| \rightarrow 0$ as $c \rightarrow \infty$. The key estimate that we prove is the following.

Proposition 3.1. *Let $\theta = \theta(n) \in [\frac{\log n}{\sqrt{n}}, \pi]$ and set, as usual, $\sigma = \sin(\theta/2)$. For all sufficiently large n there exists a fixed constant C independent of n , θ and the non-trivial representation $\rho_{\mathbf{a}}$ such that*

$$|r_{\mathbf{a}}(\theta)|^{n(\log n + C)/\sigma^2} < d_{\mathbf{a}}^{-2}.$$

The following sum-of-dimension bound is proved in Section 4.

Proposition 3.2. *Uniformly in $c \geq 1$ and n ,*

$$\sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^{-c/\log n} = O(e^{-c/8}).$$

Combining these propositions implies Theorem 1.2.

Deduction of Theorem 1.2. The lower bound was proven by Rosenthal. For the upper bound, recall that $N = 2n + 1$ and set $t = n(\log n + c)/2\sigma^2$. By the upper bound lemma,

$$\|\delta_{\text{Id}} P_\theta^t - \nu\|_{T.V.}^2 \leq \frac{1}{4} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 |r_{\mathbf{a}}(\theta)|^{2t},$$

and substituting the bounds of Propositions 3.1 and 3.2 this is bounded by

$$\frac{1}{4} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^{(C-c)/\log n} = O(e^{-c/8}),$$

as required. □

To gain a heuristic understanding of our character bound argument, consider the case of the trivial character $\chi_{\mathbf{0}}$. Of course, $r_{\mathbf{0}}(\theta) \equiv 1$; we will analyze $r_{\mathbf{a}}(\theta)$ for other \mathbf{a} by viewing the associated integral as a perturbation of the integral for $r_{\mathbf{0}}(\theta)$.

In the special case $\mathbf{a} = \mathbf{0}$, set $\omega_j = (j - 1/2)/n$ for α_j . We aim to choose $\alpha = \omega$ in Lemma 2.1 where ω is the location of a real saddle point in the integral. In this way, the dominant part of the character ratio is given by the part of the integral very near $z = \omega$. To this end, introduce

$$g_{\mathbf{0}}(z) = \theta z - \frac{1}{n} \sum_{j=1}^n \log(z^2 + \omega_j^2) \tag{5}$$

so that (4) becomes²

$$r_{\mathbf{0}}(\theta) = \frac{(2n-1)!}{(2n \sin(\theta/2))^{2n-1}} \frac{1}{2\pi i} \int_{(\omega)} e^{ng_{\mathbf{0}}(z)} dz.$$

²The notation $\int_{(\omega)}$ indicates a contour on the line $\Re(z) = \omega$.

A saddle point in the integral occurs for ω solving

$$g'_0(\omega) = \theta - \frac{1}{n} \sum_{j=1}^n \frac{2\omega}{\omega^2 + \omega_j^2} = 0. \quad (6)$$

The information that we will need regarding g_0 and its first few derivatives near the saddle point is contained in the following lemma, whose proof we postpone to the end of this section.

Lemma 3.3. *For $0 < \theta < \pi - \frac{(\log n)^2}{n}$, the saddle point ω is given by*

$$\omega = (1 + O(\log^{-2} n)) \cot\left(\frac{\theta}{2}\right).$$

For fixed $\theta \in (0, \pi)$,

$$\omega = (1 + O_\theta(n^{-1})) \cot\left(\frac{\theta}{2}\right).$$

Also, in the same range of θ we have the asymptotics (g_0 from (5))

$$g_0^{(2)}(\omega) = \frac{2}{1 + \omega^2} (1 + O(n^{-1}(1 \wedge \omega^{-2}))), \quad g_0^{(3)} \sim \frac{2\omega}{(1 + \omega^2)^2},$$

and for real t satisfying $|t| < \omega \vee \frac{1}{2}$, the bound

$$|g_0^{(4)}(\omega + it)| \leq 40(1 \wedge \omega^{-4}).$$

In particular, for all n sufficiently large,

$$\Re \left(g_0(\omega) - g_0 \left(\omega + \frac{i\sqrt{1 + \omega^2}}{3} \right) \right) \geq \frac{1}{40}.$$

Finally, for any $\omega > 0$ and $t \geq 0$,

$$\Re g_0(\omega + i(t + 1/n)) < \Re g_0(\omega + it). \quad (7)$$

Assuming this lemma, it is a standard exercise in the saddle point method to write

$$r_0(\theta) = \frac{(2n-1)!e^{ng_0(\omega)}}{2\pi(2n \sin \theta/2)^{2n-1}} \left\{ \int_{|t| < \frac{1\vee\omega}{\log n}} + \int_{\frac{1\vee\omega}{\log n} < |t| < 1\vee\omega} + \int_{|t| > 1\vee\omega} \right\} e^{ng_0(\omega+it) - ng_0(\omega)} dt.$$

In the first integral, we Taylor expand the exponent as

$$-ng_0^{(2)}(\omega) \frac{t^2}{2} + ing_0^{(3)}(\omega) \frac{t^3}{6} + O \left(n|t|^4 \sup_{|s| < \frac{1\vee\omega}{\log n}} |g_0^{(4)}(\omega + is)| \right).$$

Thus the first integral is equal to $(1 + O(1/n)) \sqrt{\frac{2\pi}{ng_0^{(2)}(\omega)}}$ (Taylor expand the higher order terms in the exponential and plug in Gaussian moments). Near the boundary $t = \pm \frac{1\vee\omega}{\log n}$, the integrand is bounded in size by $e^{-n \log^{-2} n(1+o(1))}$; using this bound and the fact that the integrand decreases in every period of length $\frac{1}{n}$ (see (7) above),

we may easily bound the second integral. For the third, use this and note that for $|t| > \omega \vee 1$, the integrand decreases by factors of e^{-cn} between dyadic intervals. Thus we may express

$$1 = r_{\mathbf{0}}(\theta) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{e^{ng_{\mathbf{0}}(\omega)}}{\sqrt{2\pi n g_{\mathbf{0}}^{(2)}(\omega)}} (1 + O(1/n)). \quad (8)$$

Notice that the same reasoning as above allows us to replace the integrand with its absolute value.

Lemma 3.4. *We have*

$$1 + O(1/n) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{ng_{\mathbf{0}}(\omega+it)}| dt. \quad (9)$$

In view of this lemma we may bound the character ratios $|r_{\mathbf{a}}(\theta)|$ by bounding the real difference

$$\Re(g_{\mathbf{a}}(\omega+it) - g_{\mathbf{0}}(\omega+it)).$$

The error of size $O(1/n)$ is too large for low dimensional representations $\rho_{\mathbf{a}}$ where $r_{\mathbf{a}}(\theta) = 1 - O(1/n)$, but for moderate and large representations, these bounds are given in Sections 5.2 and 5.3. The small representations are treated in Section 5.1 using a differential rather than integral method.

Proof of Lemma 3.3. Write

$$\begin{aligned} g'_{\mathbf{0}}(z) &= \theta - \frac{1}{n} \sum_{j=1}^n \left[\frac{1}{z + i \frac{j-1/2}{n}} + \frac{1}{z - i \frac{j-1/2}{n}} \right] \\ &= \theta + i(\psi(n+1/2 - inz) - \psi(-n+1/2 - inz)) \\ &= \theta + i \left(\psi \left(n + \frac{1}{2} - inz \right) - \psi \left(n + \frac{1}{2} + inz \right) + \pi \tan(i\pi n z) \right) \end{aligned}$$

where ψ is the dilogarithm $\psi(z) = \frac{\Gamma'}{\Gamma}(z)$. Suppose $z = \omega > 0$ is real. Then

$$i\pi \tan(i\pi n\omega) = -\pi + O(e^{-\pi n\omega}),$$

while $\psi(x) = \log x - 1/2x + O(x^{-2})$ gives

$$\begin{aligned} &i \left(\psi \left(n + \frac{1}{2} - in\omega \right) - \psi \left(n + \frac{1}{2} + in\omega \right) \right) \\ &= 2 \tan^{-1} \left(\frac{\omega}{1 + 1/2n} \right) + \frac{1}{n} \frac{\omega}{1 + \omega^2} + O \left(\frac{1 \wedge \omega^{-2}}{n^2} \right). \end{aligned} \quad (10)$$

Thus the first claim follows from

$$\frac{\theta}{2} = \cot^{-1} \left(\frac{\omega}{1 + 1/2n} \right) + O \left(\frac{1 \wedge \omega^{-1}}{n} \right) + O(e^{-\pi n\omega}).$$

We may express higher derivatives of g_0 in terms of the polygammas $\psi_k(z) = \frac{d^k}{dz^k} \psi(z)$:

$$\begin{aligned} g_0^{(2)}(z) &= n \left(\psi_1 \left(n + \frac{1}{2} - inz \right) + \psi_1 \left(n + \frac{1}{2} + inz \right) - \pi^2 \sec^2(i\pi n z) \right) \\ g_0^{(3)}(z) &= -in^2 \left(\psi_2 \left(n + \frac{1}{2} - inz \right) - \psi_2 \left(n + \frac{1}{2} + inz \right) + 2\pi^3 \tan \cdot \sec^2(i\pi n z) \right) \\ g_0^{(4)}(z) &= -n^3 \left(\psi_3 \left(n + \frac{1}{2} - inz \right) + \psi_3 \left(n + \frac{1}{2} + inz \right) + 2\pi^4 (\sec^4 - 2 \tan^2 \sec^2)(i\pi n z) \right). \end{aligned}$$

On $\Re(z) = \omega$, the terms involving \sec are exponentially small, so may be ignored. All of the claims follow from

$$\psi_1(z) = z^{-1} + O(|z|^{-2}), \quad \psi_2(z) = -z^{-2} + O(|z|^{-3}), \quad \psi_3(z) = 2z^{-3} + O(|z|^{-4});$$

we will just check the explicit bound for $g_0^{(4)}$:

$$\begin{aligned} g_0^{(4)}(\omega + it) &\sim 2 \left[(1 + t - i\omega)^{-3} + (1 - t + i\omega)^{-3} \right] \\ &= 4 \left[\frac{1 + 3(t - i\omega)^2}{(1 - (t - i\omega)^2)^3} \right] = 4 \left[\frac{1 + 3t^2 - 3\omega^2 - 6it\omega}{(1 - t^2 + \omega^2 + 2it\omega)^3} \right]. \end{aligned}$$

Now $|1 - t^2 + \omega^2 + 2it\omega|^2 = 1 + 2\omega^2 + \omega^4 + t^4 + 2\omega^2 t^2 - 2t^2$. When $\omega \geq 1/2$, $t \leq \omega$ and $|1 - t^2 + \omega^2 + 2it\omega|^2 \geq (1 + \omega^4)$, so that

$$|g_0^{(4)}(\omega + it)| \leq 4 \frac{1 + 9\omega^2}{(1 + \omega^4)^{3/2}} \leq 40(1 \wedge \omega^{-4}).$$

When $\omega < 1/2$, $t \leq 1/2$, and therefore $|1 - t^2 + \omega^2 + 2it\omega|^2 \geq 1/2$ while $|t - i\omega|^2 \leq \frac{1}{2}$. Thus in this case $g_0^{(4)} \leq 4 \times \frac{5}{2} \times 2^{\frac{3}{2}} < 40$.

We may estimate $\Re \left(g_0(\omega) - g_0 \left(\omega + \frac{i\sqrt{1+\omega^2}}{3} \right) \right)$ by Taylor expansion about ω .

Finally, to prove the last claim, note that

$$\Re(g_0(\omega + i(t + 1/n)) - g_0(\omega + it)) = \log \left| \frac{\omega + i \left(t - \frac{2n-1}{2n} \right)}{\omega + i \left(t + \frac{2n+1}{2n} \right)} \right| < 0.$$

□

4 Bounds for dimensions

This section collects together the basic estimates for dimensions of representations that we will need throughout the mixing time upper bound argument. We conclude by proving Proposition 3.2.

Since $d_0 = 1$, the dimension equation (2) may be written as³

$$d_{\mathbf{a}} = \prod_{k=1}^n \frac{\tilde{a}_k}{k - 1/2} \prod_{1 \leq j < k \leq n} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k - 1/2)^2 - (j - 1/2)^2}.$$

³Recall $\tilde{a}_j = a_j + j - 1/2$.

We may treat this product incrementally as

$$d_{\mathbf{a}} = \prod_{k=1}^n d_{\mathbf{a}}(k), \quad d_{\mathbf{a}}(k) = \frac{\tilde{a}_k}{k-1/2} \prod_{1 \leq j < k} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k-1/2)^2 - (j-1/2)^2}.$$

If we replace \tilde{a}_j with $j-1/2 \leq \tilde{a}_j$ in the product above, then we find

$$d_{\mathbf{a}}(k) \leq \frac{(a_k + 2k - 1)!}{a_k!(2k - 1)!}. \quad (11)$$

In particular, this gives the following weak estimate for the dimension.

Lemma 4.1. *Recall the notation $\alpha_k = \tilde{a}_k/n$. We have, for each k ,*

$$\log d_{\mathbf{a}}(k) \leq a_k \log \left(1 + \frac{2k-1}{a_k}\right) + (2k-1) \log \left(1 + \frac{a_k}{2k-1}\right) + O(1).$$

In particular,

$$\log d_{\mathbf{a}} \leq O(n^2) + 2n \sum_{k: \alpha_k \geq 1} \log \alpha_k.$$

Proof. The first line follows from Stirling's approximation, and the second follows on summing. \square

To get more refined estimates, in later sections we will frequently make index-shift arguments, and so it is most convenient to attach to representation $\rho_{\mathbf{a}}$ the shift-index \mathbf{s} given by

$$\mathbf{s} = (s_1, \dots, s_n), \quad s_1 = a_1, \quad s_i = a_i - a_{i-1}, \quad i = 2, \dots, n.$$

Note that as \mathbf{a} runs over non-zero weakly increasing strings of length n , \mathbf{s} runs over all non-zero strings in \mathbb{N}^n , and \mathbf{a} is recovered from \mathbf{s} as $a_j = \sum_{i \leq j} s_i$. We abuse notation by writing $d_{\mathbf{s}} = d_{\mathbf{a}}$ for \mathbf{s} associated to \mathbf{a} .

Treating the product $d_{\mathbf{s}}$ now as a whole, we may split it as $d_{\mathbf{s}} = d_{\mathbf{s}}^- d_{\mathbf{s}}^+ d_{\mathbf{s}}^0$, where

$$\begin{aligned} d_{\mathbf{s}}^- &= \prod_{1 \leq j < k \leq n} \frac{\left[\sum_{j < i \leq k} s_i\right] + k - j}{k - j} \\ d_{\mathbf{s}}^+ &= \prod_{1 \leq j < k \leq n} \frac{\left[\sum_{i \leq j} s_i + \sum_{i \leq k} s_i\right] + j + k - 1}{j + k - 1} \\ d_{\mathbf{s}}^0 &= \prod_k \frac{\left[\sum_{i \leq k} s_i\right] + k - 1/2}{k - 1/2} \end{aligned}$$

Note that $d_{\mathbf{s}}^* \geq 1$.

Let \mathbf{e}_i denote the i th standard basis vector in \mathbb{N}^n . Our first set of bounds concern dimensions of shifts supported at the standard basis vectors.

Lemma 4.2. *For any $\mathbf{s}_0 \in \mathbb{N}^{n-i}$ we have*

$$\frac{d_{(\mathbf{0}_i, \mathbf{s}_0) + \mathbf{e}_{i+1}}}{d_{(\mathbf{0}_i, \mathbf{s}_0)}} \leq d_{\mathbf{e}_{i+1}}.$$

Proof. This is immediate from comparing separately each factor d_s^- , d_s^+ and d_s^0 . \square

We can easily bound $d_{\mathbf{e}_i}$.

Lemma 4.3. *Let $m = \min(i - 1, n - i + 1)$. We have the bound*

$$\log d_{\mathbf{e}_i} \leq m \left[1 + \log 2 + \log \frac{n}{m} + \log \frac{n}{i - 1/2} \right] + 2(n - i)[1 + \log 2] + \log \frac{n}{i - 1/2}.$$

In particular, $\log d_{\mathbf{e}_i} = O(n)$.

Proof. Splitting $d_{\mathbf{se}_i} = d_{\mathbf{se}_i}^- d_{\mathbf{se}_i}^+ d_{\mathbf{se}_i}^0$ as above, we bound each in turn.

$$d_{\mathbf{se}_i}^- = \prod_{1 \leq j < i} \prod_{i \leq k \leq n} \frac{s + (j - k)}{j - k} \leq \exp \left(s \sum_{1 \leq j < i} \sum_{i \leq k \leq n} \frac{1}{j - k} \right),$$

so dividing according to $j - k \leq m$ and $j - k > m$ below, we have

$$\frac{\log d_{\mathbf{se}_i}^-}{s} \leq \sum_{1 \leq j < i} \sum_{i \leq k \leq n} \frac{1}{j - k} \leq m + m \left[\frac{1}{m + 1} + \dots + \frac{1}{n - 1} \right] \leq m \left[1 + \log \frac{n}{m} \right].$$

Similarly,

$$\begin{aligned} d_{\mathbf{se}_i}^+ &= \prod_{1 \leq j < i} \prod_{i \leq k \leq n} \frac{s + j + k - 1}{j + k - 1} \cdot \prod_{i \leq j < k \leq n} \frac{2s + j + k - 1}{j + k - 1} \\ &\leq \exp \left(s \left(\sum_{1 \leq j < i} \sum_{i \leq k \leq n} \frac{1}{j + k - 1} + 2 \sum_{i \leq j < k \leq n} \frac{1}{j + k - 1} \right) \right) \end{aligned}$$

so that

$$\begin{aligned} \frac{\log d_{\mathbf{se}_i}^+}{s} &\leq \sum_{1 \leq j < i} \sum_{i \leq k \leq n} \frac{1}{j + k - 1} + 2 \sum_{i \leq j < k \leq n} \frac{1}{j + k - 1} \\ &\leq m \left[\frac{1}{i} + \frac{1}{i + 1} + \dots + \frac{1}{2n - 1} \right] + 2 \left[\frac{1}{2i} + \frac{2}{2i + 1} + \dots + \frac{n - i}{n + i - 1} \right] \\ &\quad + 2(n - i) \left[\frac{1}{n + i} + \frac{1}{n + i + 1} + \dots + \frac{1}{2n - 1} \right] \\ &\leq m \log \frac{2n}{i - 1/2} + 2(n - i) [1 + \log 2]. \end{aligned}$$

Finally,

$$d_{\mathbf{se}_i}^0 = \prod_{k \geq i} \frac{s + k - 1/2}{k - 1/2} \leq \exp \left(s \sum_{k \geq i} \frac{1}{k - 1/2} \right)$$

so

$$\frac{\log d_{\mathbf{se}_i}^0}{s} \leq \log \frac{n}{i - 1/2}.$$

\square

As a consequence of this lemma we get a crude bound for the dimension of any representation.

Corollary 4.4. *For all sufficiently large n , each representation $\rho_{\mathbf{a}}$ has dimension bounded by*

$$d_{\mathbf{a}} \leq e^{10na_n}.$$

If $a_i = 0$ for all $i \leq n/2$ then

$$d_{\mathbf{a}} \leq \exp(|\mathbf{a}|(\log n + 7)).$$

Proof. Note that in shift notation, $a_n = |\mathbf{s}| = \sum_j s_j$. By Lemma 4.2,

$$d_{\mathbf{s}} \leq \prod_{i=1}^n d_{s_i \mathbf{e}_i}.$$

The maximum over i of the upper bound for $s^{-1} \log d_{s_i \mathbf{e}_i}$ in Lemma 4.3 is less than $10n$ for all n sufficiently large. Combining these observations, the first claim of the corollary follows.

To prove the second, observe that $|\mathbf{a}| = \sum_i (n - i + 1)s_i$. For all $i > n/2$ Lemma 4.3 gives $\log d_{s_i \mathbf{e}_i} \leq (n - i + 1)s_i(\log n + 7)$, which suffices. \square

The preceding bounds are useful when we perform the right-side shifts before the left ones. Sometimes we will wish to perform left-side shifts first. The following lemma gives bounds in this case.

Lemma 4.5. *Let $\mathbf{s} \in \mathbb{N}^j$. Let $m = \min(j, n - j + 1)$ and let $1 \leq \eta \leq m$ be a parameter. Write*

$$|\mathbf{s}|_{\text{loc}} = \sum_{j-\eta \leq i \leq j} s_i.$$

Then we have the bound

$$\begin{aligned} \log \frac{d_{(\mathbf{s}, \mathbf{0}) + \mathbf{e}_j}}{d_{(\mathbf{s}, \mathbf{0})}} &\leq m \left[\log \frac{n + |\mathbf{s}|_{\text{loc}}}{m + |\mathbf{s}|_{\text{loc}}} + \log \frac{n + j}{m + j} + 2 \right] \\ &\quad + \eta \log(n - j + \eta) + 2(n - j + 1) + \log \frac{n + j}{m + j} + O(1). \end{aligned}$$

Proof. With error at most $O(1)$, we have

$$\begin{aligned} \log \frac{d_{(\mathbf{s}, \mathbf{0}) + \mathbf{e}_j}^-}{d_{(\mathbf{s}, \mathbf{0})}^-} &= \log \left[\prod_{1 \leq i < j} \prod_{j \leq k \leq n} \left(1 + \frac{1}{k - i + \sum_{i < \ell \leq j} s_\ell} \right) \right] \\ &\leq \sum_{j-\eta \leq i < j} \sum_{j \leq k \leq n} \frac{1}{k - i} + \sum_{i \leq j-\eta} \sum_{j \leq k \leq n} \frac{1}{k - i + |\mathbf{s}|_{\text{loc}}} \\ &\leq \eta \log(n - j + \eta) + m \left(1 + \log \frac{n + |\mathbf{s}|_{\text{loc}}}{m + |\mathbf{s}|_{\text{loc}}} \right). \quad (\text{see (12) below}) \end{aligned}$$

Meanwhile,

$$\begin{aligned} \log \frac{d_{(\mathbf{s}, \mathbf{0}) + \mathbf{e}_j}^+}{d_{(\mathbf{s}, \mathbf{0})}^+} &\leq \log \frac{d_{(\mathbf{0}, \mathbf{0}) + \mathbf{e}_j}^+}{d_{(\mathbf{0}, \mathbf{0})}^+} \\ &\leq \log \left[\prod_{1 \leq i < j} \prod_{j \leq k \leq n} \left(1 + \frac{1}{i + k - 1} \right) \cdot \prod_{j \leq i < k \leq n} \left(1 + \frac{1}{i + k - 1} \right) \right] \end{aligned}$$

The first of these terms is bounded by

$$\frac{1}{j} + \frac{2}{j+1} + \dots + \frac{m}{j+m-1} + m \left[\frac{1}{j+m} + \dots + \frac{1}{n+j-1} \right] \leq m \left[1 + \log \frac{n+j}{m+j} \right]. \quad (12)$$

The second is bounded by

$$\frac{1}{2j} + \frac{2}{2j+1} + \dots + \frac{n-j+1}{n+j} + (n-j+1) \left[\frac{1}{n+j} + \dots + \frac{1}{2n+2j-1} \right] \leq 2(n-j+1).$$

Finally,

$$\log \frac{d_{(\mathbf{s}, \mathbf{0}) + \mathbf{e}_j}^0}{d_{(\mathbf{s}, \mathbf{0})}^0} = \log \left[\prod_{k \geq j} \left(1 + \frac{1}{k - 1/2 + |\mathbf{s}|} \right) \right] \leq \log \frac{n - 1/2}{j - 1/2}.$$

□

It will also be convenient to have some lower bounds for dimensions.

Lemma 4.6. *Let η , $1 \leq \eta \leq n$ be a parameter, and set $\mathbb{N}^n = \mathbb{N}^{n-\eta} \oplus \mathbb{N}^\eta$. We have*

$$d_{(\mathbf{s}_1, \mathbf{s}_2)} \geq d_{(\mathbf{s}_1, \mathbf{0})}^+ d_{(\mathbf{0}, \mathbf{s}_2)}^-.$$

Set $|\mathbf{s}_1| = \sum_{i \leq n-\eta} s_i$. We have

$$d_{(\mathbf{s}_1, \mathbf{0})}^+ \geq \left(1 + \frac{|\mathbf{s}_1|}{2n} \right)^{n\eta - \eta^2}.$$

In particular, for $|\mathbf{s}_1| < n$, and for $\eta < n^{1-\epsilon}$ and all sufficiently large n ,

$$d_{(\mathbf{s}_1, \mathbf{0})}^+ \geq e^{|\mathbf{s}_1|/\eta^3}.$$

Define $|\mathbf{s}_2|_p = \sum_{i < \eta} ((\eta - i + 1)s_i)$. Then

$$d_{(\mathbf{0}, \mathbf{s}_2)}^- \geq \prod_{1 \leq j < n-\eta} \prod_{1 \leq k \leq \eta} \left(1 + \frac{\sum_{i \leq k} s_i}{\eta + j} \right) \geq \prod_{1 \leq j < n-\eta} \left(1 + \frac{|\mathbf{s}_2|_p}{\eta + j} \right).$$

Proof. The identity $d_{(\mathbf{s}_1, \mathbf{s}_2)} \geq d_{(\mathbf{s}_1, \mathbf{0})}^+ d_{(\mathbf{0}, \mathbf{s}_2)}^-$ is immediate from the definitions.

Now we have

$$\begin{aligned} d_{(\mathbf{s}_1, \mathbf{0})}^+ &= \prod_{1 \leq j < k \leq n} \left(\frac{[\sum_{i \leq j} s_i + \sum_{i \leq k} s_i] + j + k - 1}{j + k - 1} \right) \\ &\geq \prod_{j < n-\eta} \prod_{n-\eta \leq k \leq n} \left(\frac{j + k - 1 + |\mathbf{s}_1|}{j + k - 1} \right) \geq \left(1 + \frac{|\mathbf{s}_1|}{2n} \right)^{n\eta - \eta^2}, \end{aligned}$$

proving the bound for $d_{(\mathbf{s}_1, \mathbf{0})}^+$ for sufficiently large n . Meanwhile, denoting s_i the components of \mathbf{s}_2 ,

$$\begin{aligned} d_{(\mathbf{0}, \mathbf{s}_2)}^- &\geq \prod_{1 \leq j < n-\eta} \prod_{1 \leq k \leq \eta} \left(1 + \frac{\sum_{i \leq k} s_i}{k + n - \eta - j} \right) \\ &\geq \prod_{1 \leq j < n-\eta} \prod_{1 \leq k \leq \eta} \left(1 + \frac{\sum_{i \leq k} s_i}{\eta + j} \right) \quad (\text{switching } j \leftrightarrow n - \eta - j) \\ &\geq \prod_{1 \leq j < n-\eta} \left(1 + \frac{\sum_{1 \leq k \leq \eta} \sum_{i \leq k} s_i}{\eta + j} \right) = \prod_{1 \leq j < n-\eta} \left(1 + \frac{|\mathbf{s}_2|_p}{\eta + j} \right). \end{aligned}$$

□

We conclude the section by proving Proposition 3.2

Proof of Proposition 3.2. Set $\eta = \lfloor n^{3/4} \rfloor$. Then letting $\mathbb{N}^n = \mathbb{N}^{n-\eta} \oplus \mathbb{N}^\eta$ observe $d_{(\mathbf{s}_1, \mathbf{s}_2)} \geq d_{(\mathbf{s}_1, \mathbf{0})}^+ d_{(\mathbf{0}, \mathbf{s}_2)}^-$, $d_{\mathbf{0}} = 1$, and therefore

$$\begin{aligned} \sum_{\mathbf{0} \neq \mathbf{s} \in \mathbb{N}^n} d_{\mathbf{s}}^{-c/\log n} &= -1 + \sum_{\mathbf{s}} d_{\mathbf{s}}^{-c/\log n} \\ &\leq -1 + \left(\sum_{\mathbf{s}_1 \in \mathbb{N}^{n-\eta}} \left(d_{(\mathbf{s}_1, \mathbf{0})}^+ \right)^{-c/\log n} \right) \cdot \left(\sum_{\mathbf{s}_2 \in \mathbb{N}^\eta} \left(d_{(\mathbf{0}, \mathbf{s}_2)}^- \right)^{-c/\log n} \right), \end{aligned}$$

so it will suffice to show that each sum on the right is $1 + O(e^{-c})$ as $n \rightarrow \infty$.

We first handle the sum over \mathbf{s}_1 . Observe that $\#\{\mathbf{s} \in \mathbb{N}^n : |\mathbf{s}| = j\} \leq n^j \wedge j^n$. Applying the bound of Lemma 4.6,

$$\begin{aligned} \sum_{\mathbf{s}_1 \in \mathbb{N}^\eta} \left(d_{(\mathbf{s}_1, \mathbf{0})}^+ \right)^{-c/\log n} &\leq 1 + \sum_{j=1}^{\infty} \#\{\mathbf{s} \in \mathbb{N}^{n-\eta} : |\mathbf{s}| = j\} \left(1 + \frac{j}{2n} \right)^{-cn^{7/4}/\log n} \\ &\leq 1 + \sum_{j=1}^{4n} n^j e^{-c'jn^{3/4}/\log n} + \sum_{j=4n}^{\infty} j^n \left(\frac{2j}{2n} \right)^{-cn^{7/4}/\log n} \\ &= 1 + o(1), \quad n \rightarrow \infty. \end{aligned}$$

Now we consider the sum over \mathbf{s}_2 . Again applying the bound of the lemma,

$$\sum_{\mathbf{s}_2 \in \mathbb{N}^\eta} \left(d_{(\mathbf{0}, \mathbf{s}_2)}^- \right)^{-c/\log n} \leq 1 + \sum_{q=1}^{\infty} \#\{\mathbf{s} \in \mathbb{N}^\eta : |\mathbf{s}|_p = q\} \left[\prod_{j < n-\eta} \left(1 + \frac{q}{\eta + j} \right) \right]^{-c/\log n}.$$

Evidently $\#\{\mathbf{s} \in \mathbb{N}^\eta : |\mathbf{s}|_p = q\}$ is the number of partitions of q into parts of size at most η . This is bounded by the total number of partitions of q , which is $e^{O(\sqrt{q})}$, and

by q^n . Thus the above sum over q is bounded by

$$\begin{aligned} & \sum_{1 \leq q < \eta/2} \exp \left(O(\sqrt{q}) - \frac{cq(\log n - \log \eta)}{2 \log n} \right) \\ & + \sum_{q=\eta/2}^{n^{5/4}} \exp (O(\sqrt{q}) - c\eta/8) \\ & + \sum_{q=n^{5/4}}^{\infty} \exp (\eta \log q - c(n - \eta) \log q / \log n); \quad \left(\frac{q}{n} \geq \log q \text{ for } n \geq 16 \right) \end{aligned}$$

the first term is $O(e^{-c/8})$ and the remaining terms are $o(1)$ as $n \rightarrow \infty$. □

5 Deterministic θ : Theorem 1.2

The heart of Theorem 1.2 is the character ratio bound in Proposition 3.1. The proof of this proposition is split into three cases depending on the size of the indices of the representations; in the next section we treat the small characters via a differential formulation of the integral formula (4). The remaining characters will be bounded directly from the integral formula, using saddle point analysis.

5.1 Character ratios of small representations

We consider a representation to be small if $\sum_i a_i \leq \frac{n}{\sigma \log n}$. The differential form of the character formula is as follows.

Lemma 5.1. *Let $\rho_{\mathbf{a}}$, $\mathbf{a} = (a_1, \dots, a_n)$ index a representation of $SO(2n+1)$. Set $m = a_n$. Recall that we define $\tilde{a}_j = a_j + j - 1/2$. Let $\tilde{b}_1 > \tilde{b}_2 > \dots > \tilde{b}_m$ complement $\{\tilde{a}_1, \dots, \tilde{a}_n\}$ in $\{1/2, 3/2, \dots, m + n - 1/2\}$. Then the character ratio at \mathbf{a} is given by*

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n+2m-1)!(2 \sin \frac{\theta}{2})^{2n-1}} \left(\prod_{k=1}^m (\tilde{b}_k^2 + \partial_{\theta}^2) \right) \left(2 \sin \frac{\theta}{2} \right)^{2n+2m-1}. \quad (13)$$

Proof. Since we know that $\chi_{\mathbf{0}}(\theta) = 1$ and this holds uniformly for all θ and all orthogonal groups $SO(2m+1)$, we obtain the integral identity

$$1 = \frac{(2m-1)!}{(2 \sin \frac{\theta}{2})^{2m-1}} \oint \frac{\sin(z\theta)}{\prod_{j=1}^m [(j-1/2)^2 - z^2]} dz,$$

valid for $m = 0, 1, 2, \dots$, $\theta \in \mathbb{T}$.

Now taking any contour that encloses the real axis between $\pm(m+n-1/2)$ we have

$$\begin{aligned} r_{\mathbf{a}}(\theta) &= \frac{(2n-1)!}{(2 \sin \frac{\theta}{2})^{2n-1}} \oint \frac{\sin(z\theta)}{\prod_{j=1}^n (\tilde{a}_j^2 - z^2)} dz \\ &= \frac{(2n-1)!}{(2 \sin \frac{\theta}{2})^{2n-1}} \oint \frac{\left(\prod_{j=1}^m (\tilde{b}_j^2 + \partial_{\theta}^2) \right) \sin(z\theta)}{\prod_{j=1}^{n+m} [(j-1/2)^2 - z^2]} dz. \end{aligned}$$

Passing the differential operator $\prod_{j=1}^m (\tilde{b}_j^2 + \partial_{\theta}^2)$ outside the integral, we obtain the required expression. □

We next give an expression for the character ratio in rising powers of $\sigma = \sin \frac{\theta}{2}$.

Lemma 5.2. *Keep the notations of the previous lemma. We have*

$$r_{\mathbf{a}}(\theta) = \sum_{s=0}^m \frac{(-4)^s (2n-1)!}{(2(n+s)-1)!} E_s \sigma^{2s} \quad (14)$$

where

$$E_s = \sum_{1 \leq j_1 < j_2 < \dots < j_s \leq m} \prod_{i=1}^s \left(\left(m + n + i - j_i - \frac{1}{2} \right)^2 - \tilde{b}_{j_i}^2 \right).$$

Remark 3. Note that $m + n + \frac{1}{2} - j - \tilde{b}_j \geq 1$ and is increasing in j . In particular, each term in each E_s is positive.

Proof. We have

$$(\tilde{b}_k^2 + \partial_\theta^2) \sigma^{2r-1} = (r-1/2)(r-1) \sigma^{2r-3} + (\tilde{b}_k^2 - (r-1/2)^2) \sigma^{2r-1}.$$

Iterating this in (13) we obtain

$$\begin{aligned} & \prod_{k=1}^m (\tilde{b}_k^2 + \partial_\theta^2) \sigma^{2n+2m-1} \\ &= \sum_{S \subseteq [m]} \frac{(2(n+m)-1)!}{(2(n+|S|)-1)!} 2^{2|S|-2m} \sigma^{2(n+|S|)-1} \prod_{j \in S} [\tilde{b}_j^2 - (n+m-m_S(j)-1/2)^2] \end{aligned} \quad (15)$$

where $m_S(j) = \#([j-1] \setminus S)$. The claim now follows on grouping terms according to $s = |S|$. \square

As an example of the previous two lemmas, we now calculate the character ratio of the first few representations. These calculations will be used in the proof of the lower bound of Theorem 1.1 in the next section.

Example 1. The trivial representation $\rho_{\mathbf{0}}$ has dimension 1 and character ratio 1.

The first non-trivial representation is the natural representation $\rho_{(\mathbf{0},1)}$. It's dimension is $2n+1$. Using the previous lemma, we may easily calculate it's character ratio. We have $m=1$ and $\tilde{b}_1 = n-1/2$ so that $E_1 = 2n$. Thus

$$r_{(\mathbf{0},1)}(\theta) = 1 - \frac{4\sigma^2}{2n+1}. \quad (16)$$

The tensor product $\rho_{(\mathbf{0},1)} \otimes \rho_{(\mathbf{0},1)}$ decomposes as a direct sum of the trivial representation, the adjoint square and the symmetric square:

$$\rho_{(\mathbf{0},1)} \otimes \rho_{(\mathbf{0},1)} = \rho_{\mathbf{0}} \oplus \rho_{(\mathbf{0},1,1)} \oplus \rho_{(\mathbf{0},2)}.$$

The adjoint square $\rho_{(\mathbf{0},1,1)}$ is a representation of dimension $n(2n+1)$. For this representation, $m=1$ and $\tilde{b}_1 = n-3/2$ so that $E_1 = 4n-2$. Thus it's character ratio is given by

$$r_{(\mathbf{0},1,1)}(\theta) = 1 - \frac{4(2n-1)\sigma^2}{n(2n+1)}. \quad (17)$$

The symmetric square $\rho_{(0,2)}$ is a representation of dimension $n(2n+3)$. For this representation, $m = 2$ and $\tilde{b}_1 = n + 1/2$, $\tilde{b}_2 = n - 1/2$. Thus $E_1 = 4n + 2$ and $E_2 = 8n^2 + 12n + 4$. Thus the character ratio is given by

$$r_{(0,2)}(\theta) = 1 - \frac{4\sigma^2}{n} + \frac{16\sigma^4}{n(2n+3)} \quad (18)$$

Lemma 5.3. *We have the exact evaluation*

$$E_1 = \sum_{i=1}^n a_i(a_i + 2i - 1) \quad (19)$$

and the bounds

$$|E_1| \leq (m+2n) \sum a_n, \quad |E_s| \leq s|E_1| \left[(3m+2n) \sum a_n \right]^{s-1}.$$

In particular, for \mathbf{a} satisfying $\sum a_i \leq \frac{n}{\sigma \log n}$ we have

$$\log r_{\mathbf{a}}(\theta) = -\frac{E_1 \sigma^2}{n^2} (1 + O(1/\log n)). \quad (20)$$

Proof. Since $\{\tilde{a}_i\}_{i=1}^n$ and $\{\tilde{b}_i\}_{i=1}^m$ form a partition of $\{1/2, 3/2, \dots, m+n-1/2\}$ we have

$$E_1 = \sum_{i=1}^m ((m+n+1/2-i)^2 - \tilde{b}_i^2) = \sum_{i=1}^n (\tilde{a}_i^2 - (i-1/2)^2).$$

Since $\tilde{a}_i = a_i + i - 1/2$ the evaluation of E_1 follows.

The bound for $|E_1|$ is immediate, since $a_i + 2i - 1 < a_n + 2n = m + 2n$.

To bound $|E_s|$, split off the sum over j_1 to obtain

$$\begin{aligned} E_s &= \sum_{j_1=1}^m \left((m+n+\frac{1}{2}-j_1)^2 - \tilde{b}_{j_1}^2 \right) \\ &\quad \times \sum_{j_1 < j_2 < \dots < j_s \leq m} \prod_{i=2}^s \left(\left(m+n+i-j_i - \frac{1}{2} \right)^2 - \tilde{b}_{j_i}^2 \right). \end{aligned}$$

In the inner sum bound

$$\tilde{b}_{j_i} + m + n + i - j_i - \frac{1}{2} \leq 3m + 2n,$$

and

$$\frac{m+n+i-j_i-\frac{1}{2}-\tilde{b}_{j_i}}{m+n+\frac{1}{2}-j_i-\tilde{b}_{j_i}} \leq i.$$

which follows from $m+n+\frac{1}{2}-j-\tilde{b}_j \geq 1$.

Therefore

$$\begin{aligned} E_s &\leq \sum_{j_1=1}^m ((m+n+\frac{1}{2}-j_1)^2 - \tilde{b}_{j_1}^2) (3m+2n)^{s-1} s! \sum_{j_1 < \dots < j_s \leq m} \prod_{i=2}^s (m+n+\frac{1}{2}-j_i - \tilde{b}_{j_i}) \\ &\leq s \left[(3m+2n) \sum_{j=1}^m (m+n+\frac{1}{2}-j - \tilde{b}_j) \right]^{s-1} \cdot E_1. \end{aligned}$$

This proves the second bound.

To prove the final claim, substitute the bounds above into (14) and use

$$\frac{\sigma^2}{n^2}(3m+2n) \sum a_i = O\left(\frac{1}{\log n}\right).$$

□

Proof of Proposition 3.1 for small representations. When $\sum a_i \leq \frac{n}{\sigma \log n}$, the above lemmas reduce the proof of Proposition 3.1 to the estimate (now independent of θ)

$$\frac{E_1(\mathbf{a})}{2n} > \frac{\log d_{\mathbf{a}}}{\log n}. \quad (21)$$

This will be most convenient to verify in the shift notation $\mathbf{s} \leftrightarrow \mathbf{a}$, $a_i = \sum_{j \leq i} s_j$. Given a shift index \mathbf{s} write $\mathbf{s} = \sum_{i=1}^n s_i \mathbf{e}_i$ for its decomposition in standard basis vectors. It is immediate from the expression (19) for E_1 that

$$E_1(\mathbf{s}) \geq \sum_{i=1}^n E_1(s_i \mathbf{e}_i).$$

Since Lemma 4.2 guarantees that $\log d_{\mathbf{s}} \leq \sum_{i=1}^n \log d_{s_i \mathbf{e}_i}$ we have reduced to checking the claim for \mathbf{s} of form $s \mathbf{e}_i$.

Now

$$E_1(s \mathbf{e}_i) = \sum_{j=i}^n s(s+2j-1) \geq s \sum_{j=i}^n 2j = s(n^2 + n + i - i^2).$$

Inserting the dimension bound for $d_{s \mathbf{e}_i}$ in Lemma 4.3 we see that we have reduced to checking

$$\frac{n^2 + n + i - i^2}{2n} \geq \min(i, n-i) \left[\frac{\log \frac{n}{\min(i, n-i)}}{\log n} + \frac{\log \frac{n}{i}}{\log n} \right] + O\left(\frac{n-i}{\log n}\right)$$

Once n is sufficiently large, this may be seen to hold for all i by considering separately the cases $n-i \ll \log n$, $\log n \ll n-i \ll \sqrt{n}$ and $\sqrt{n} \ll n-i \ll n$. □

5.2 Character ratios of moderate representations

In this section we extend the proof of Proposition 3.1 to representations $\rho_{\mathbf{a}}$ satisfying $a_n \leq \frac{2 \cdot 10^6 n}{\sigma}$, $\sigma = \sin(\frac{\theta}{2})$. Since we have treated the case $\sum a_j \leq \frac{n}{\sigma \log n}$ in the section on small representations, we may assume that this no longer holds. In this section we will make use of the assumption that $\theta > \frac{\log n}{\sqrt{n}}$ and we will further assume that $\pi - \theta \geq \frac{\log^2 n}{n}$. The case in which θ is closer to π is treated in Section 5.4.

As a point of reference, we record the following lemma regarding the dimensions of moderate representations.

Lemma 5.4. *Let \mathbf{a} satisfy $a_n \leq \frac{2 \cdot 10^6 n}{\sigma}$ and $\sum a_j \geq \frac{n}{\sigma \log n}$. Then for all sufficiently large n , we have the bounds*

$$\exp\left(\frac{n(5 - \log \sigma)}{4 \cdot 10^6 \log n}\right) \leq d_{\mathbf{a}} \leq \exp\left(\frac{2 \cdot 10^7 n^2}{\sigma}\right).$$

Proof. The upper bound is immediate from Corollary 4.4.

For the lower bound, write

$$d_{\mathbf{a}} \geq d_{\mathbf{a}}^+ = \prod_{j>i} \frac{a_j + a_i + j + i - 1}{j + i - 1} \geq \prod_{j>i} \left(1 + \frac{a_i + a_j}{2n}\right).$$

The conditions guarantee that for $C := 2 \cdot 10^6 / \sigma$, $\frac{a_i + a_j}{2n} < C$, so that $1 + \frac{a_i + a_j}{2n} \geq \exp(\frac{(a_i + a_j) \log C}{2Cn})$. Thus

$$d_{\mathbf{a}} \geq \prod_{j>i} \exp\left(\frac{(a_i + a_j) \log C}{2Cn}\right) \geq \exp\left(\frac{(n-1) \log C}{Cn} \sum a_i\right) \geq \exp\left(\frac{n(5 - \log \sigma)}{4 \cdot 10^6 \log n}\right)$$

□

In analogy with (5) for the trivial character, introduce (recall $\alpha_j = \tilde{a}_j/n$)

$$g_{\mathbf{a}}(z) = \theta z - \frac{1}{n} \sum_{j=1}^n \log(z^2 + \alpha_j^2), \quad (22)$$

so that, fixing the line of integration at the saddle point $\omega \sim \cot(\frac{\theta}{2})$ for the trivial representation,

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{1}{2\pi i} \int_{(\omega)} e^{ng_{\mathbf{a}}(z)} dz. \quad (23)$$

Heuristically, since

$$\frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{ng_{\mathbf{0}}(\omega+it)}| dt \doteq 1$$

we may bound

$$|r_{\mathbf{a}}(\theta)| \leq \sup_t \left| \frac{e^{ng_{\mathbf{a}}(\omega+it)}}{e^{ng_{\mathbf{0}}(\omega+it)}} \right|.$$

In practice, we put in the sup bound only for small $|t| \ll \frac{1}{\sigma \sqrt{\log n}}$ and rely on the rapid decay of the integrand in t to take care of the rest of the integral.

Before we begin we introduce the length function

$$\ell(x; t, \omega)^2 = |\omega + it - ix|^2 |\omega + it + ix|^2 = (x^2 - (t^2 - \omega^2))^2 + 4\omega^2 t^2. \quad (24)$$

This function plays a prominent role in the next two sections, e.g. because we may write

$$\Re[g_{\mathbf{a}}(\omega + it) - g_{\mathbf{0}}(\omega + it)] = \frac{1}{n} \sum_{j=1}^n (\log \ell(\omega_j; t, \omega) - \log \ell(\alpha_j; t, \omega)).$$

We record several of its simple properties.

Lemma 5.5. *Let t and $\omega > 0$ be fixed and consider ℓ as a function of x only. If $|t| < \omega$ then ℓ is minimized at $x = 0$ with minimum $t^2 + \omega^2$. If $|t| \geq \omega$ then ℓ is minimized at $|x| = \sqrt{t^2 - \omega^2}$ with minimum $2|t|\omega$. In the case $|t| > \omega$, for $0 < \delta < \sqrt{t^2 - \omega^2}$ we have*

$$\ell(\sqrt{t^2 - \omega^2} - \delta) < \ell(\sqrt{t^2 - \omega^2} + \delta).$$

We now work to prove our supremum bound for small t . Our goal will be the estimate

$$\frac{n^2(\log n + C)}{2\sigma^2} \Re [g_{\mathbf{a}}(\omega + it) - g_{\mathbf{0}}(\omega + it)] \leq -\log d_{\mathbf{a}} \quad (25)$$

for a sufficiently large fixed constant C . As in the small representation section, our argument is based upon making shifts to the index, but whereas in that section we shifted the rightmost indices first, here we shift indices from left to right.

Lemma 5.6. *Let $\mathbf{s} \in \mathbb{N}^j$ and recall that we set \mathbf{e}_j for the j th standard unit vector. For t satisfying $t \leq 10^4 \sqrt{\frac{1+\omega^2}{\log n}}$,*

$$\begin{aligned} \Re [g_{(\mathbf{s}, \mathbf{0})+\mathbf{e}_j}(\omega + it) - g_{(\mathbf{s}, \mathbf{0})}(\omega + it)] \leq \\ -\frac{1}{2n} \frac{(1 - \frac{j-1}{n})(1 + \frac{j+|\mathbf{s}|}{n})((1 + \frac{|\mathbf{s}|}{n})^2 + (\frac{j+|\mathbf{s}|}{n})^2 + 2\omega^2)}{((1 + \frac{|\mathbf{s}|}{n})^2 + \omega^2)^2} \left(1 - \frac{4t^2}{1 + \omega^2} + O(1/n)\right). \end{aligned} \quad (26)$$

Remark 4. For reference, note that if $j \approx n$ and $|\mathbf{s}|$ is much smaller than $n(1 + \omega^2)$ then the RHS of the above bound is roughly $-\frac{2(1-\frac{j}{n})}{n(1+\omega^2)}$.

Proof. Let \mathbf{a} correspond to $(\mathbf{s}, 0)$. Then $\alpha_j = \frac{j+|\mathbf{s}|-1/2}{n}$ and $\alpha_n = 1 + \frac{|\mathbf{s}|-1/2}{n}$. We have

$$\begin{aligned} \Re [g_{(\mathbf{s}, \mathbf{0})+\mathbf{e}_j}(\omega + it) - g_{(\mathbf{s}, \mathbf{0})}(\omega + it)] &= \frac{1}{n} \log \frac{\ell(\alpha_j)}{\ell(\alpha_n + \frac{1}{n})} \\ &= \frac{1}{2n} \log \frac{(\alpha_j^2 + \omega^2 - t^2)^2 + 4\omega^2 t^2}{((\alpha_n + 1/n)^2 + \omega^2 - t^2)^2 + 4\omega^2 t^2} \\ &= \frac{1}{2n} \log \left[1 - \frac{((\alpha_n + 1/n)^2 - \alpha_j^2)((\alpha_n + 1/n)^2 + \alpha_j^2 + 2\omega^2 - 2t^2)}{((\alpha_n + 1/n)^2 + \omega^2 - t^2)^2 + 4\omega^2 t^2} \right] \\ &\leq -\frac{1}{2n} \frac{((\alpha_n + 1/n)^2 - \alpha_j^2)((\alpha_n + 1/n)^2 + \alpha_j^2 + 2\omega^2 - 2t^2)}{((\alpha_n + 1/n)^2 + \omega^2 + t^2)^2} \end{aligned}$$

Since $\alpha_n + 1/n \geq 1$, the result follows on substituting the values of α_j and α_n and factoring out expressions involving t . \square

Lemma 5.7. *Let $(\mathbf{s}, \mathbf{0})$, $\mathbf{s} \in \mathbb{N}^j$, be the shift index of \mathbf{a} , and let $z = \omega + it$ with $|t| \leq 10^4 \sqrt{\frac{1+\omega^2}{\log n}}$.*

(i) *If $a_n = |\mathbf{s}| \leq \frac{n\sqrt{1+\omega^2}}{\log n}$ then for sufficiently large fixed C ,*

$$\frac{1}{2} n^2 (\log n + C) (1 + \omega^2) \cdot \Re [g_{(\mathbf{s}, \mathbf{0})+\mathbf{e}_j}(z) - g_{(\mathbf{s}, \mathbf{0})}(z)] \leq -\log \frac{d_{(\mathbf{s}, \mathbf{0})+\mathbf{e}_j}}{d_{(\mathbf{s}, \mathbf{0})}}. \quad (27)$$

(ii) *In the range $\frac{n\sqrt{1+\omega^2}}{\log n} \leq |\mathbf{s}| \leq 2 \cdot 10^6 n \sqrt{1+\omega^2}$ there is a fixed constant C' such that if $\omega > C'$ then the bound (27) continues to hold.*

(iii) *For $\omega < C'$ there is a third fixed constant $C'' > 0$ such that*

$$\frac{1}{2} n^2 (\log n + C) (1 + \omega^2) \cdot \Re [g_{(\mathbf{s}, \mathbf{0})+\mathbf{e}_j}(z) - g_{(\mathbf{s}, \mathbf{0})}(z)] \leq -C'' (n - j + 1) \log n.$$

Each of C, C' and C'' is independent of n, θ , and \mathbf{s} . In particular,

(iv) The bound (27) holds unless $n - j + 1 \leq n^\kappa$ for a fixed universal $\kappa < 1$.

Proof. (i) The restrictions on $|\mathbf{s}|$ and t allow us to write (26) as

$$\Re \left[g_{(\mathbf{s}, \mathbf{0}) + \mathbf{e}_j}(z) - g_{(\mathbf{s}, \mathbf{0})}(z) \right] \leq - \frac{(1 - \frac{j-1}{n})(1 + \frac{j}{n})(1 + \frac{j^2}{n^2} + 2\omega^2)}{2n(1 + \omega^2)^2} (1 + O(1/\log n)).$$

The relative error term may evidently be ignored by choosing the constant C sufficiently large. Set $m = n - j + 1$. For all j the RHS above is less than

$$- \frac{2}{(1 + \omega^2)n^2 \log n} \cdot m \log n \cdot \max((1 - O(m/n)), c)$$

for some fixed $c > 0$. Meanwhile, we have a bound of $\geq -m \log(n/m) + O(m)$ for the RHS of (27) by taking $\eta = 1$ in Lemma 4.5. The first claim follows by choosing C sufficiently large to cover the case of small m .

(ii) For $|\mathbf{s}| \leq 400n\sqrt{1 + \omega^2}$ if ω is larger than a sufficiently large fixed constant then in fact the net effect of the $\frac{|\mathbf{s}|}{n}$ terms in the RHS of (26) is negative, so that the above argument goes through without further restriction on $|\mathbf{s}|$.

(iii) In any case, in the range $|\mathbf{s}| \leq 400n\sqrt{1 + \omega^2}$, inclusion of the factors of $\frac{|\mathbf{s}|}{n}$ changes the RHS of (26) by at most a constant factor, which proves the third claim.

(iv) The claim regarding κ now follows, since uniformly in $|\mathbf{s}| \leq 400n\sqrt{1 + \omega^2}$ we have that the LHS of (27) is less than $-c'm \log n$, for a fixed $c' > 0$, while the RHS is $\geq -m \log(n/m) + O(m)$, with $m = n - j + 1$ as before. \square

Lemma 5.7 allow us to prove Proposition 3.1 for moderate representations that satisfy $a_n \ll \frac{n}{\sigma \log n}$. While the strict increment inequality (27) does not necessarily hold for every increment in the range $a_n \ll \frac{n}{\sigma}$, the next lemma demonstrates that it holds for ‘most’ increments, which suffices for our purpose.

Lemma 5.8. *Let \mathbf{a} satisfy $a_n \leq 2 \cdot 10^6 n \sqrt{1 + \omega^2}$ and let $z = \omega + it$ with $|t| \leq 10^4 \sqrt{\frac{1 + \omega^2}{\log n}}$. Then for C the large fixed constant of the previous lemma and for n sufficiently large,*

$$\frac{1}{2} n^2 (\log n + C + 1) (1 + \omega^2) \cdot \Re [g_{\mathbf{a}}(z) - g_{\mathbf{0}}(z)] \leq -\log d_{\mathbf{a}}. \quad (28)$$

Proof. Let \mathbf{s} correspond to \mathbf{a} and set $|\mathbf{s}| = a_n = m$. Using standard basis vectors we may write (in accordance with left-to-right shift)

$$\mathbf{s} = \sum_{j=1}^m \mathbf{e}_{i_j}, \quad i_1 \leq i_2 \leq \dots \leq i_m;$$

set also $\mathbf{s}^j = \sum_{k=1}^j \mathbf{e}_{i_k}$ with $\mathbf{s}^0 = \mathbf{0}$. Obviously

$$\begin{aligned} & \frac{1}{2} n^2 (\log n + C + 1) (1 + \omega^2) \cdot \Re [g_{\mathbf{a}}(z) - g_{\mathbf{0}}(z)] + \log d_{\mathbf{a}} \\ &= \sum_{j=1}^m \left[\frac{1}{2} n^2 (\log n + C + 1) (1 + \omega^2) \cdot \Re [g_{\mathbf{s}^j}(z) - g_{\mathbf{s}^{j-1}}(z)] + \log \frac{d_{\mathbf{s}^j}}{d_{\mathbf{s}^{j-1}}} \right]. \end{aligned} \quad (29)$$

If either $m < \frac{n\sqrt{1+\omega^2}}{\log n}$ or $\omega > C'$ then we may apply either (i) or (ii) of the previous lemma to conclude that each term in the sum is negative so that we are done. So we may assume that m is large and that ω is bounded.

Call $k = \left\lfloor \frac{n\sqrt{1+\omega^2}}{\log n} \right\rfloor$ and let $m_0 > k$ denote the index of the first positive term in the sum. Note that $i_{m_0} \geq n - n^\kappa$ for some fixed $\kappa < 1$, by (iv) of Lemma 5.7.

We first argue that we may assume that $m - m_0$ is large by noting that the first k terms in the sum of (29) are substantially negative. Recall that ω is assumed to be bounded. Applying Lemma 4.6 with $\eta = n - i_k + 1$, we deduce for n sufficiently large that

$$d_{\mathbf{s}^k} \geq \exp(k(n - i_k + 1)/3) \geq \exp\left(\frac{n(n - i_k + 1)}{3 \log n}\right).$$

Since the sum up to k in (29) is negative, even when $C + 1$ is replaced by C , it follows by comparing with $\frac{\log d_{\mathbf{s}^k}}{\log n}$ that

$$\begin{aligned} & \frac{1}{2}n^2(\log n + C + 1)(1 + \omega^2) \cdot \Re[g_{\mathbf{s}^k}(z) - g_0(z)] + \log d_{\mathbf{s}^k} \\ & \leq (1 + o(1)) \frac{-n(n - i_k + 1)}{3(\log n)^2}. \end{aligned} \quad (30)$$

Now for $j > k$, Lemma 4.5 gives that

$$\log \frac{d_{\mathbf{s}^j}}{d_{\mathbf{s}^{j-1}}} \leq (n - i_j + 1)(\log n + O(1)) \leq (n - i_k + 1)(\log n + O(1)), \quad (31)$$

and so we immediately obtain that (29) is negative unless $m - m_0 \geq \frac{n}{4(\log n)^3}$.

Let $\delta > 0$ be a small, fixed, positive constant and set

$$J = \left\lceil \frac{\log(n - i_{m_0} + 1)}{\log(1 + \delta)} \right\rceil \ll \log n.$$

We partition $\{i_{m_0}, i_{m_0+1}, \dots, i_m\}$ into J sets by defining

$$S_j = \{i_\lambda : i_\lambda \in n - I_j\}, \quad I_j = [(1 + \delta)^{j-1}, (1 + \delta)^j), \quad j = 1, 2, \dots, J.$$

We perform a trimming on the sets S_j . Let $M = \frac{2n}{3J(\log n)^4} \gg \frac{n}{(\log n)^5}$. We discard all S_j with $|S_j| < M$. From each remaining set we form S'_j by removing the smallest $M/2$ of the i_λ from S_j . Altogether we have discarded at most $\frac{3}{2}JM \leq \frac{n}{(\log n)^4}$ of the i_λ . Now in view of (31), the total contribution to (29) of the discarded i_λ is bounded by

$$(1 + o(1))(n - i_k + 1) \frac{n}{(\log n)^3},$$

which is negligible; see (30).

We now claim that if δ was chosen to be appropriately small, then for each remaining $i_\lambda \in S'_j$,

$$\frac{1}{2}n^2(\log n + C + 1)(1 + \omega^2) \cdot \Re[g_{\mathbf{s}^\ell}(z) - g_{\mathbf{s}^{\ell-1}}(z)] \leq -\log \frac{d_{\mathbf{s}^\ell}}{d_{\mathbf{s}^{\ell-1}}}.$$

Setting $\mathbf{s} = \mathbf{s}^{\ell-1}$, $j = i_\lambda$ and $\eta = 3\delta(n - i_\lambda + 1)$ in Lemma 4.5 we find that $|\mathbf{s}|_{\text{loc}} \geq M/2$, (it includes all of the deleted points from the set containing i_λ)

$$\log \frac{d_{\mathbf{s}^\ell}}{d_{\mathbf{s}^{\ell-1}}} \leq 3\delta(n - i_k + 1) \log n + O((n - i_k + 1) \log_2 n).$$

Choosing 3δ sufficiently smaller than C'' from the previous lemma proves the claim and finishes the proof. \square

Lemma 5.8 completes our supremum bound for small t , so we now turn to bounding the tail of the integral in (23).

Lemma 5.9. *Introduce the function*

$$m_\omega(t) = \max_{\mathbf{a}: a_n \leq 2 \cdot 10^6 n \sqrt{1+\omega^2}} \Re(g_{\mathbf{a}}(\omega + it)). \quad (32)$$

This satisfies the following properties.

1. *The maximum in $m_\omega(t)$ is achieved at $\mathbf{a} = \mathbf{0}$ when $t^2 \leq \omega^2 + 1/2$.*
2. *The monotonicity property $m_\omega(t) \geq m_\omega(t + \frac{1}{n})$ holds for all $t \geq 0$.*
3. *For $t > 4 \cdot 10^6 \sqrt{1+\omega^2}$, there is a $c > 0$ such that $m_\omega(2t) \leq m_\omega(t) - cn$.*

Proof. Recall

$$\Re(g_{\mathbf{a}}(\omega + it)) = \omega\theta - \frac{1}{n} \sum_j \log(\ell(\alpha_j)).$$

In view of the last claim of Lemma 5.5, the minimizing choice for \mathbf{a} is $\mathbf{0}$ when the minimum for $\ell(x; t, \omega)$ occurs for $|x| \leq \frac{1}{2}$. This proves the first claim.

Notice, also, that the optimal choice for the α_j will be a contiguous block, i.e. the optimizing \mathbf{a} is of form $(k)^n$ for some $k \geq 0$. This suffices to prove the second claim, since if $(k)^n$ achieves the maximum in $m_\omega(t + \frac{1}{n})$ then

$$m_\omega(t) \geq \Re(g_{(k-1)^n}(\omega + it)) > \Re(g_{(k)^n}(\omega + i(t + 1/n))) = m_\omega(t + 1/n),$$

while if $\mathbf{0}$ achieves the maximum in $m_\omega(t + \frac{1}{n})$ then

$$m_\omega(t) \geq \Re(g_{\mathbf{0}}(\omega + it)) > \Re(g_{\mathbf{0}}(\omega + i(t + 1/n))) = m_\omega(t + 1/n),$$

by applying the last claim of Lemma 3.3.

Finally, notice that the restriction on a_n is equivalent to $\alpha_n \leq 1 + 2 \cdot 10^6 \sqrt{1+\omega^2}$. For all $|t| > 4 \cdot 10^6 \sqrt{1+\omega^2}$ the maximizing \mathbf{a} is easily seen to be the block with a_n as large as possible. The last claim now follows from Euclidean geometry. \square

We now bound the tail of the integral for $r_{\mathbf{a}}(\theta)$.

Lemma 5.10. *We have the bound*

$$\int_{|t| > 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} |e^{ng_{\mathbf{a}}(\omega+it)}| dt \leq \exp \left(ng_{\mathbf{0}}(\omega) - \frac{(10^8 + o(1))n}{\log n} \right).$$

Proof. In the integral we may evidently replace $g_{\mathbf{a}}(\omega+it)$ with $m_\omega(t)$. When $|t| < 1 \vee \omega$, $m_\omega(t) = \Re(g_{\mathbf{0}}(\omega + it))$; for $|t| = 10^4 \sqrt{\frac{1+\omega^2}{\log n}} + O(1/n)$ Taylor expansion of $g_{\mathbf{0}}$ around ω gives [Lemma 3.3]

$$\Re g_{\mathbf{0}}(\omega + it) \leq g_{\mathbf{0}}(\omega) - \frac{10^8 + o(1)}{\log n}.$$

The bound now follows easily on applying the monotonicity of m_ω and rapid decay for $|t| > 4 \cdot 10^6 \sqrt{1+\omega^2}$. \square

We now put together our pair of bounds to prove Proposition 3.1 for moderate representations.

Proof of Proposition 3.1. Recall that (8) of Section 3 gives

$$1 = r_{\mathbf{0}}(\theta) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{e^{ng_{\mathbf{0}}(\omega)}}{\sqrt{2\pi n g_{\mathbf{0}}^{(2)}(\omega)}} (1 + O(1/n))$$

while (9) of Section 3 gives

$$1 + O(1/n) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{ng_{\mathbf{0}}(\omega+it)}| dt.$$

Thus we have the bound

$$\begin{aligned} |r_{\mathbf{a}}(\theta)|(1 - O(1/n)) &\leq \sup_{|t| \leq 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} \exp(n\Re(g_{\mathbf{a}}(\omega+it) - g_{\mathbf{0}}(\omega+it))) \\ &\quad + \sqrt{2\pi g_{\mathbf{0}}^{(2)}(\omega)n} \int_{|t| > 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} \exp(n\Re(g_{\mathbf{a}}(\omega+it) - g_{\mathbf{0}}(\omega))) dt. \end{aligned} \quad (33)$$

By Lemmas 5.8 and 5.10 the RHS is bounded by

$$\leq \exp\left(-\frac{2\sigma^2}{n(\log n + O(1))} \log d_{\mathbf{a}}\right) + O\left(\exp\left(\frac{-10^8 n(1+o(1))}{\log n}\right)\right).$$

Now we appeal to the dimension bounds for moderate representations that we proved in Lemma 5.4 at the beginning of this section. Since $d_{\mathbf{a}} \leq \exp(\frac{2 \cdot 10^7 n^2}{\sigma})$, the last expression is

$$\leq (1 + o(1/n)) \exp\left(-\frac{2\sigma^2}{n(\log n + O(1))} \log d_{\mathbf{a}}\right).$$

We have thus shown that

$$\log |r_{\mathbf{a}}(\theta)| \leq \frac{-2\sigma^2}{n \log n} \log d_{\mathbf{a}} (1 + O(1/\log n)) + O(1/n).$$

Since $d_{\mathbf{a}} \geq \exp\left(\frac{n(5-\log \sigma)}{4 \cdot 10^6 \log n}\right)$ and $\sigma \geq \frac{\log n}{\sqrt{n}}$ it follows that in fact

$$\log |r_{\mathbf{a}}(\theta)| \leq \frac{-2\sigma^2}{n \log n} \log d_{\mathbf{a}} (1 + O(1/\log n))$$

which proves Proposition 3.1 for moderate representations. \square

5.3 Large representations

Among those large representations, for which $a_n > \frac{2 \cdot 10^6 n}{\sigma}$, we distinguish further two kinds. Let $k = n - \lfloor \frac{n}{2\sqrt{1+\omega^2}} \rfloor$. If $a_k < \frac{10^6 k}{\sigma}$ we say that $\rho_{\mathbf{a}}$ has ‘controlled growth’. Otherwise we say that $\rho_{\mathbf{a}}$ is ‘giant’.

5.3.1 Controlled growth

In the case that $\rho_{\mathbf{a}}$ has controlled growth, let $k < m < n$ be maximal such that $a_m < \frac{2 \cdot 10^6 m}{\sigma}$. We are going to view $r_{\mathbf{a}}(\theta)$ as a perturbation of the character ratio $r_{\mathbf{a}(m)}(\theta)$ on $SO(2m+1)$, where $\mathbf{a}(m)$ denotes \mathbf{a} truncated at a_m .

Let, as usual, ω denote the location of the saddle point in the integral of the character ratio for the trivial representation on $SO(2n+1)$, and let ω' solving $g'_{\mathbf{0}(m)}(\omega') = 0$ denote the corresponding saddle point for $SO(2m+1)$. Our first lemma controls the discrepancy between these two saddle points.

Lemma 5.11. *Keep, as usual, the restrictions $\frac{\log n}{\sqrt{n}} < \theta < \pi - \frac{(\log n)^2}{n}$. For all sufficiently large n ,*

$$|\omega - \omega'| \leq \omega \left(\frac{1}{m} - \frac{1}{n} \right) \leq \frac{1}{n}.$$

Also, for all α between ω and ω' and for all $|t| \leq 1/2 \vee \alpha$,

$$|\Re(g'_{\mathbf{0}}(\alpha + it))| = O\left(\frac{|t| + \frac{1}{n}}{1 \vee \omega}\right).$$

Proof. In view of (10) in the proof of Lemma 3.3,

$$g'_{\mathbf{0}(m)}(\omega) = |g'_{\mathbf{0}(m)}(\omega) - g'_{\mathbf{0}}(\omega)| \leq \frac{2\omega}{1 + \omega^2} \left[\frac{1}{m} - \frac{1}{n} \right] + O(n^{-2}(1 + \omega^2)^{-1}).$$

The first claim then follows, since $g_{\mathbf{0}(m)}^{(2)}(\alpha) \sim \frac{2}{1 + \alpha^2}$ for all $\omega < \alpha < \omega'$.

Now $g'_{\mathbf{0}}(\alpha + it) = g'_{\mathbf{0}}(\alpha + it) - g'_{\mathbf{0}}(\omega)$. Since $|\alpha - \omega| \leq \frac{1}{n}$, the bound follows from the bounds for derivatives of $g_{\mathbf{0}}$ in Lemma 3.3. \square

We now choose a contour in the integral formula for $r_{\mathbf{a}}(\theta)$ passing through ω' and given by

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \frac{1}{2\pi i} \left\{ \int_{|\Im(z)| \leq 10^4 \sqrt{\frac{1+\omega^2}{\log m}}}^{\Re(z)=\omega'} + \int_{\mathcal{C}} \right\} e^{ng_{\mathbf{a}}(z)} dz.$$

Here \mathcal{C} is a contour that depends on ω' : if $\omega' \geq 1/200$ then the contour consists of just $\{\omega' + it : |t| \geq 10^4 \sqrt{\frac{1+\omega^2}{\log n}}\}$. For $\omega' < 1/200$ the contour has four pieces:

$$\begin{aligned} \mathcal{C} = & \left\{ \omega' + it : 10^4 \sqrt{\frac{1+\omega^2}{\log n}} \leq |t| \leq \frac{\sqrt{1+\omega^2}}{3} \right\} \cup \left\{ \alpha \pm \frac{i\sqrt{1+\omega^2}}{3} : \alpha \in \left[\omega', \frac{1}{200} \right] \right\} \\ & \cup \left\{ \frac{1}{200} + it : |t| > \frac{\sqrt{1+\omega^2}}{3} \right\}. \end{aligned}$$

This particular choice of contour allows us to put in a sup-bound for the contributions of a_j , $j > m$ in the integrand by ensuring that the contour does not pass too near any associated $\alpha_j = \tilde{a}_j/n$.

In fact, throughout this section we are going to argue by keeping track of the incremental change to the integrand $e^{ng_{\mathbf{a}(m)}(z)}$ and to $d_{\mathbf{a}(m)}$ as we append successively

each a_j , $j > m$. We have already discussed the information that we need regarding the dimension increment at the beginning of Section 4. Recall that there we introduced $d_{\mathbf{a}}(j)$ such that

$$d_{\mathbf{a}} = \prod_{j=1}^n d_{\mathbf{a}}(j), \quad d_{\mathbf{a}(m)} = \prod_{j=1}^m d_{\mathbf{a}}(j).$$

For those $j > m$, Lemma 4.1 gives for each $j > m$ that

$$\log d_{\mathbf{a}}(j) \leq (2j - 1) \log \alpha_j + O(n).$$

As in the section on moderate representations, we are going to put a sup bound in for the integrand $e^{ng_{\mathbf{a}}(\omega' + it)}$ when $|t|$ is small, and then give a more trivial bound for the tail contour \mathcal{C} . The sup bound for small $|t|$ is as follows.

Lemma 5.12. *For $|t| < 10^4 \sqrt{\frac{1+\omega^2}{\log m}}$ and for a sufficiently large fixed constant C ,*

$$n\Re(g_{\mathbf{a}}(\omega' + it) - g_{\mathbf{0}}(\omega + it)) \leq -\frac{2\sigma^2 \log d_{\mathbf{a}}}{n(\log n + C)}. \quad (34)$$

Proof. In the stated range of $|t|$ and for α between ω and ω' we have $g'_{\mathbf{0}}(\alpha + it) = o(1)$. Since $|\omega - \omega'| = O(1/n)$ it follows that

$$ng_{\mathbf{0}(m)}(\omega' + it) = ng_{\mathbf{0}(m)}(\omega + it) + o(1).$$

Thus we may write the LHS of (34) as

$$m\Re(g_{\mathbf{a}(m)}(\omega' + it) - g_{\mathbf{0}(m)}(\omega' + it)) - \sum_{j=m+1}^n \Re \log \frac{\alpha_j^2 + (\omega' + it)^2}{\left(\frac{j-1/2}{n}\right)^2 + (\omega' + it)^2} + o(1).$$

Since $\rho_{\mathbf{a}(m)}$ is either a small or a moderate representation of $SO(2m+1)$,

$$m\Re(g_{\mathbf{a}(m)}(\omega' + it) - g_{\mathbf{0}(m)}(\omega' + it)) \leq -\frac{2\sigma^2 \log d_{\mathbf{a}(m)}}{m(\log m + C)},$$

and therefore the proof is completed on observing that

$$o(1) + \sum_{j=m+1}^n \left[\frac{2\sigma^2 \log d_{\mathbf{a}}(j)}{n(\log n + C)} - \Re \log \frac{\alpha_j^2 + (\omega' + it)^2}{\left(\frac{j-1/2}{n}\right)^2 + (\omega' + it)^2} \right] < 0$$

in view of the bound for $d_{\mathbf{a}}(j)$ above, and using $\alpha_j \geq 10^6 \sqrt{1 + \omega^2}$ for all $m < j \leq n$. \square

Lemma 5.13. *Keep the definitions of m and C from above and let $m < j \leq n$. We have*

$$\inf_{z \in \mathcal{C}} |z^2 + \alpha_j^2| \geq 2(1/200 \vee \omega)\alpha_j$$

Proof. Observe that on any line $\Re(z) = \alpha$, the minimum of $|z^2 - (i\alpha_j)^2|$ is $4\alpha^2\alpha_j^2$. It only remains to check that for $\omega < 1/200$, the minimum of $|(\omega + it)^2 - (i\alpha_j)^2|$ for $t < \sqrt{1 + \omega^2}/3$ exceeds $2\alpha_j/200$, but this is obvious geometrically. \square

We now bound the integral over the contour \mathcal{C} .

Lemma 5.14. *We have the bound*

$$\int_{z \in \mathcal{C}} \exp(n\Re g_{\mathbf{a}}(z) - ng_0(\omega)) d|z| \leq \exp\left(\frac{(-10^8 + o(1))m}{\log m}\right) \prod_{j=m+1}^n \left(\frac{\sqrt{1+\omega^2}}{\alpha_j}\right)^{1/10}.$$

Proof. In view of the last lemma, the integral is evidently bounded by

$$\prod_{j=m+1}^n \frac{1+\omega^2}{2\alpha_j(1/200 \vee \omega)} \times \int_{z \in \mathcal{C}} \exp(\Re[\theta(n-m)(z-\omega) + mg_{\mathbf{a}(m)}(z) - mg_0(m)(\omega)]) d|z|.$$

Using $\frac{1}{2(a \max b)} \leq \frac{1}{\sqrt{a^2+b^2}}$, each term in the product is bounded by $\frac{200\sqrt{1+\omega^2}}{\alpha_j} \leq \left(\frac{\sqrt{1+\omega^2}}{\alpha_j}\right)^{1/10}$.

As regards the integral, if $\omega' > 1/200$ then the result now follows from Lemma 5.10 of the previous section. If $\omega' \leq 1/200$ Lemma 5.10 still bounds the vertical part of the integral that is nearest the real axis, so it remains to bound the horizontal part, and the vertical part that extends to $\pm i\infty$. Taylor expanding $g_0(m)$ at $\omega' + i\sqrt{1+\omega'^2}/3$ we find

$$g_0(m)(\omega' + i\sqrt{1+\omega'^2}/3) - g_0(m)(\omega) \leq -1/40 + o(1).$$

Using $\theta < \pi$ and $n-m < m$ we find that throughout the horizontal part of \mathcal{C} , where $|t| = \sqrt{1+\omega^2}/3$, the integrand is bounded by

$$\exp(-(1/40 - \pi/200)m + o(m)) < \exp(-m/200).$$

Therefore the remainder of the integral contributes $\exp(-m/200 + o(m))$ by mimicking the proof of Lemma 5.10. \square

Proof of Proposition 3.1 . Arguing as in the proof for moderate representations,

$$\begin{aligned} (1 - O(1/n))|r_{\mathbf{a}}(\theta)| &\leq \sup_{|t| \leq 10^4 \sqrt{\frac{1+\omega^2}{\log m}}} \exp(n\Re(g_{\mathbf{a}}(\omega' + it) - g_0(\omega + it))) \\ &\quad + \sqrt{2\pi n g_0^{(2)}(\omega)} \int_{z \in \mathcal{C}} \exp(n\Re g_{\mathbf{a}}(z) - ng_0(\omega)) d|z| \end{aligned}$$

Putting in the bounds of Lemmas 5.12 and 5.14, the RHS is

$$\leq \exp\left(-\frac{2\sigma^2 \log d_{\mathbf{a}}}{n(\log n + C)}\right) + \exp\left(\frac{(-10^8 + o(1))m}{\log m}\right) \prod_{j=m+1}^n \left(\frac{\sqrt{1+\omega^2}}{\alpha_j}\right)^{1/10}.$$

As in the earlier proof,

$$\exp\left(\frac{(-10^8 + o(1))m}{\log m}\right) = o(1/m) \exp\left(-\frac{2\sigma^2 \log d_{\mathbf{a}(m)}}{n(\log n + C)}\right).$$

Moreover,

$$\exp\left(-\frac{2\sigma^2 \sum_{j=m+1}^n \log d_{\mathbf{a}}(j)}{n(\log n + C)}\right) \geq \prod_{j=m+1}^n \left(\frac{\sqrt{1+\omega^2}}{\alpha_j}\right)^{1/10}$$

for all n sufficiently large. Thus

$$\log |r_{\mathbf{a}}(\theta)| \leq O(1/n) - \frac{2\sigma^2 \log d_{\mathbf{a}}}{n(\log n + C)}.$$

The error term of size $O(1/n)$ is dealt with as before. \square

5.3.2 Giant representations

Once the representation is giant, trivial considerations suffice to bound the character ratio. We take the integral contour on the line $\Re(z) = \omega \vee 1$ so as to avoid nearby poles of the integrand, and put in a sup bound for all but the first factor from the product, reserving the first factor to ensure convergence. This yields

$$\begin{aligned} |r_{\mathbf{a}}(\theta)| &\leq O\left(\sqrt{ng_0^{(2)}(\omega)}\right) \int_{\Re(z)=1\vee\omega} e^{n\Re(g_{\mathbf{a}}(z)-g_0(\omega))} d|z| \\ &\ll \frac{\sqrt{n}e^{\theta(1\vee\omega-\omega)n}}{\sqrt{1+\omega^2}} \int \left| \frac{\left(\frac{1}{n}\right)^2 + \omega^2}{\alpha_1^2 + ((1\vee\omega) + it)^2} \right| dt \sup_{\Re(z)=1\vee\omega} \prod_{j=2}^n \left| \frac{\left(\frac{j-1/2}{n}\right)^2 + \omega^2}{\alpha_j^2 + z^2} \right|. \end{aligned}$$

The integral is $O(\omega \wedge \omega^2)$. If $\alpha_j < 1 \vee \omega$ then the denominator of the j th term in the product is minimized at $\Re(z) = 0$ with minimum value $\alpha_j^2 + (1 \vee \omega)^2$. Thus the j th term is bounded by 1. Otherwise, if $\alpha_j \geq 1 \vee \omega$ the minimum value of the denominator is $2\alpha_j(1 \vee \omega) > \alpha_j(1 + \omega^2)$.

Therefore, we obtain the bound

$$|r_{\mathbf{a}}(\theta)| \leq O(n^{1/2} e^{((1\vee\omega)-\omega)\theta n}) \prod_{\substack{j:\alpha_j > 1\vee\omega \\ j > 1}} \frac{1}{\alpha_j}.$$

Applying the dimension bound of Lemma 4.1, we deduce that for giant representations

$$\begin{aligned} \log |r_{\mathbf{a}}(\theta)| + \frac{2\sigma^2 \log d_{\mathbf{a}}}{n \log n} &\leq - \sum_{\substack{j:\alpha_j > 1\vee\omega \\ j > 1}} \log \alpha_j + \delta_{\omega < 1}(1 - \omega)\theta n \\ &\quad + O\left(\frac{\sigma^2 n}{\log n}\right) + \frac{2\sigma^2}{\log n} \sum_{j:\alpha_j > 1\vee\omega} \log \alpha_j \end{aligned} \tag{35}$$

We may see that this expression is asymptotically negative as follows. As regards the two sums over j for which $\alpha_j > 1 \vee \omega$, the first is asymptotically larger than the second by a factor of at least $\log n$. Moreover, for all $j > k$,

$$\alpha_j > 10^6 \sqrt{1 + \omega^2} \frac{k}{n} \geq 5 \cdot 10^5 \sqrt{1 + \omega^2}$$

so that the first term is of order at least $\Omega(n)$. Thus the two terms on the second line are asymptotically dominated.

We need compare the first two terms only for $\omega < 1$. In this case, using $\alpha_j > 5 \times 10^5$ for $j > k$, and $k \leq n(1 - 1/2\sqrt{2})$, we see that

$$\sum_{\substack{j:\alpha_j > 1\vee\omega \\ j > 1}} \log \alpha_j \geq \sum_{j=k+1}^n \log(5 \times 10^5) > 4.6n > \pi n,$$

which shows that the first term of the RHS of (35) dominates the second. This completes the proof of Proposition 3.1 for giant representations.

5.4 The case of θ close to π

Our treatment of moderate and large representations in the last two sections completes the proof Theorem 1.2 for θ varying with n in the range $\frac{\log n}{\sqrt{n}} < \theta < \pi - \frac{(\log n)^2}{n}$. The case $\theta \approx \pi$ is difficult because the location of the saddle point cannot be determined accurately. We now show that a modification of Rosenthal's argument in the case $\theta = \pi$ suffices to cover the range $\theta \geq \pi - \frac{(\log n)^2}{n}$.

For this range of θ , $\sin(\theta/2) = 1 - O(n^{-2+\epsilon})$, and so we seek the estimate

$$\log |r_{\mathbf{a}}(\theta)| \leq -\frac{2 \log d_{\mathbf{a}}}{n(\log n + O(1))}.$$

The section on small representations gives this estimate already for any \mathbf{a} satisfying $\sum_j a_j \leq \frac{n}{\theta \log n}$, so we may assume that $a_n \geq \frac{\sqrt{n}}{2 \log n}$ and $\sum_j a_j \geq \frac{n}{4(\log n)^2}$. Applying Lemma 4.6 with $\eta = 1$ in the first case, or $\eta = n - 1$ in the second, we deduce that the dimensions of the remaining representations to be considered exceed $\exp(\frac{\sqrt{n}}{3 \log n})$.

Introduce

$$\bar{r}_{\mathbf{a}}(\pi) = \frac{(2n-1)!}{2^{2n-1}} \sum_{j=1}^n \left| \frac{\sin(\tilde{a}_j \pi)}{\tilde{a}_j \prod_{r \neq j} (\tilde{a}_r^2 - \tilde{a}_s^2)} \right|.$$

Note that

$$r_{\mathbf{a}}(\theta) \leq (\sin \theta/2)^{-2n+1} \bar{r}_{\mathbf{a}}(\pi) \leq (1 + O((\log n)^2/n)) \bar{r}_{\mathbf{a}}(\pi).$$

Rosenthal's upper bound in the case $\theta = \pi$ is proven by showing that $\log \bar{r}_{\mathbf{a}}(\pi) \leq \frac{-2 \log d_{\mathbf{a}}}{n(\log n + C)}$. Therefore, bounding the remaining factor in $r_{\mathbf{a}}(\theta)$,

$$\log |r_{\mathbf{a}}(\theta)| \leq O\left(\frac{(\log n)^2}{n}\right) - \frac{2 \log d_{\mathbf{a}}}{n(\log n + C)} \leq -\frac{2 \log d_{\mathbf{a}}}{n(\log n + C)}(1 + O(n^{-1/2+\epsilon})).$$

Thus the error may be absorbed into the constant.

6 Mixture of rotations

The remainder of the paper concerns random walks wherein at each step the angle of rotation is chosen from a fixed distribution. If ξ is this probability distribution on \mathbb{T}^1 , the mixture walk P_{ξ} has Fourier coefficients

$$\xi(r_{\mathbf{a}}) = \int_{\mathbb{T}^1} r_{\mathbf{a}}(\theta) d\xi(\theta).$$

Generically we expect that the mixing time in total variation is controlled by the eigenvalue at the lowest dimensional non-trivial representation. In this case, this is the natural representation of dimension $2n + 1$, with eigenvalue

$$\xi(r_{(\mathbf{0},1)}) = \int_{\mathbb{T}^1} \left(1 - \frac{2(\sin \theta/2)^2}{n}\right) d\xi(\theta) = 1 - \frac{4\xi(\sigma^2)}{2n+1},$$

leading to a predicted total variation mixing time of

$$\frac{\log d_{(\mathbf{0},1)}}{-\log \xi(r_{(\mathbf{0},1)})} \sim \frac{n \log n}{2\xi(\sigma^2)}. \quad (36)$$

In the case of the mixture walk, the situation is muddled a bit because the quantity $\frac{\log d_{\mathbf{a}}}{-\log |\xi(r_{\mathbf{a}})|}$ is not necessarily maximized at the natural representation. Heuristically this is suggested by our Proposition 3.1, which proves the bound

$$|\xi(r_{\mathbf{a}})| \leq \int_{\mathbb{T}^1} d_{\mathbf{a}}^{\frac{-2(\sin \theta/2)^2}{n(\log n + C)}} d\xi(\theta).$$

This is only an upper bound, but for large representations $\rho_{\mathbf{a}}$ this bound suggests that $\xi(r_{\mathbf{a}})$ is largely controlled by the part of ξ nearest 0 (the issue is that the integration is not in the exponent).

In the proof that follows we confirm the natural representation prediction (36) for the mixing time, proving Theorem 1.1, but as the above discussion suggests, we do not follow the customary path of bounding the L^1 norm with the L^2 norm, instead using a truncation argument to bypass the larger dimensional representations. In the following section we prove a cut-off for the L^2 norm at a point which depends on the smallest point in the support of measure ξ , thus confirming Theorem 1.3.

6.1 Random θ in total variation: Proof of Theorem 1.1

The lower bound follows from a standard application of the second moment method (see [5]) applied to the function $\chi_{(\mathbf{0},1)}$. The necessary estimates appear in Example 1:

$$\begin{aligned} d_{(\mathbf{0},1)} &= 2n + 1, & \xi(r_{(\mathbf{0},1)}) &= 1 - \frac{2\xi(\sigma^2)}{n} + O(n^{-2}) \\ d_{(\mathbf{0},1,1)} &= n(2n + 1), & \xi(r_{(\mathbf{0},1,1)}) &= 1 - \frac{4\xi(\sigma^2)}{n} + O(n^{-2}) \\ d_{(\mathbf{0},2)} &= n(2n + 3), & \xi(r_{(\mathbf{0},2)}) &= 1 - \frac{4\xi(\sigma^2)}{n} + O(n^{-2}) \end{aligned}$$

together with the decomposition

$$\chi_{(\mathbf{0},1)}^2 = \chi_{\mathbf{0}} + \chi_{(\mathbf{0},1,1)} + \chi_{(\mathbf{0},2)}.$$

We refer the reader to Rosenthal's proof of the lower bound in the case of deterministic θ , [17] Theorem 2.1, where the details are the same.

For the upper bound, recall that we set $\mu_{\theta} = \delta_{Id} \cdot P_{\theta}$ for the generating measure of the fixed- θ walk. Proposition 3.1 guarantees that there exists a $C > 0$ such for all n sufficiently large, and for all non-trivial representations \mathbf{a} ,

$$\log |\hat{\mu}_{\theta}(\chi_{\mathbf{a}})| = \log |r_{\mathbf{a}}(\theta)| \leq -\frac{2\sigma^2(\theta)}{n(\log n + C)} \log d_{\mathbf{a}}.$$

Let $c > 0$ and let $t = \frac{n(\log n + 2C + 2c)}{2\xi(\sigma)^2}$. Conditioning on the choices of θ at each step of the walk, and applying the triangle inequality, we have

$$\|\delta_{Id} P_{\xi}^t - \nu\| \leq \int \int \int_{\theta \in (\mathbb{T}^1)^t} \|\mu_{\theta_1} \star \cdots \star \mu_{\theta_t} - \nu\| \xi(d\theta_1) \cdots \xi(d\theta_t).$$

Since the total variation distance is bounded by 1, for any measurable set $E \subset (\mathbb{T}^1)^t$ we obtain the bound

$$\|P_\xi^t \delta_{\text{Id}} - \nu\| \leq \xi^{\otimes t}(E) + \iiint_{\theta \in E^c} \left(\frac{1}{2} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 \prod_{j=1}^t |\hat{\mu}_{\theta_j}(\chi_{\mathbf{a}})|^2 \right) d\xi(\theta_1) \cdots d\xi(\theta_t),$$

by applying the Upper Bound Lemma on E^c . We now define

$$E = \left\{ \theta \in (\mathbb{T}^1)^t : \sum_{j=1}^t 2\sigma^2(\theta_j) \leq n(\log n + C + c) \right\}.$$

Then the classical Central Limit Theorem gives that there exists a constant K such that

$$\xi^{\otimes t}(E) \ll \exp\left(-\frac{c^2 n}{K(\log n)^2}\right).$$

Meanwhile, applying Proposition 3.2, the integral over E^c is bounded by

$$\sup_{\theta \in E^c} \frac{1}{2} \sum_{\mathbf{a} \neq \mathbf{0}} \exp\left(2 \log d_{\mathbf{a}} \left[1 - \frac{1}{n(\log n + C)} \sum_{j=1}^t 2\sigma^2(\theta_j)\right]\right) \leq \frac{1}{2} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^{\frac{-c}{\log n}} = O(e^{-c/8}),$$

completing the proof.

6.2 Random θ walk in L^2 : Theorem 1.3

Our proof of Theorem 1.3 exhibits a competition between the character ratios vs dimension growth for small and moderate representations. The large representations will be inconsequential. We will first prove the lower bound, which is illustrative. Then we will discuss how to modify the argument from the fixed θ setting to obtain the upper bound.

Throughout this section, it will be convenient to use as reference points block representations of the form $\mathbf{a}(s, t) = (\mathbf{0}_{n-n^t}, (n^s)_{n^t})$, $0 < s, t < 1$.⁴ As a starting point we estimate the dimensions of these representations.

Lemma 6.1. *Let $\mathbf{a}(s, t) := (\mathbf{0}_{n-n^t}, (n^s)_{n^t})$ with $0 < s, t < 1$. Also let $\mathbf{a}(u) = \mathbf{a}(u, u)$. We have*

$$\log d_{\mathbf{a}(s, t)} = (1 - s \vee t) n^{s+t} (\log n + O(1)).$$

Proof. As usual, we write $\log d_{\mathbf{a}(s, t)} = \log d_{\mathbf{a}(s, t)}^0 + \log d_{\mathbf{a}(s, t)}^+ + \log d_{\mathbf{a}(s, t)}^-$. We dispatch with the first two terms quickly, as they are of lower order.

We have

$$\log d_{\mathbf{a}(s, t)}^0 = \sum_{n-n^t < j \leq n} \log \frac{n^s + j - 1/2}{j - 1/2} = O(n^{s+t-1}).$$

Meanwhile

$$\log d_{\mathbf{a}(s, t)}^+ = \sum_{i \leq n-n^t} \sum_{n-n^t < j \leq n} \log \frac{n^s + j + i - 1}{j + i - 1} = O(n^{s+t}),$$

⁴Throughout, n^x is assumed to be an integer.

since the logarithm is $O(n^{s-1})$.

With the term $\log d_{\mathbf{a}(s,t)}^-$ we must take more care. We write this as

$$\begin{aligned} \log d_{\mathbf{a}(s,t)}^- &= \sum_{i \leq n-n^t} \sum_{n-n^t < j \leq n} \log \frac{n^s + j - i}{j - i} \\ &= \sum_{i \leq n-n^t} \sum_{j \leq n^t} \log \frac{n^s + j + i - 1}{j + i - 1} \\ &= \sum_{k=1}^{n^t} k \log \frac{k + n^s}{k} + n^t \sum_{n^t < k \leq n-n^t} \log \frac{k + n^s}{k} + \sum_{n-n^t < k \leq n} (n-k) \log \frac{k + n^s}{k}. \end{aligned}$$

The last of these sums is $O(n^{s+t})$ since the logarithm is $O(n^{s-1})$. The first is also $O(n^{s+t})$ since it is bounded by [use $\log(1+x) \leq x$]

$$\sum_{k \leq n^t} k \log \left(1 + \frac{n^s}{k} \right) \leq \sum_{k \leq n^t} n^s = n^{s+t}.$$

In the middle sum, if $s > t$ then we split the sum further at n^s . The first part of the sum becomes

$$n^t \sum_{n^t < k \leq n^s} \log \frac{k + n^s}{k} \leq O(n^{s+t}) + n^t \sum_{k \leq n^s} \log \frac{n^s}{k} = O(n^{s+t}).$$

We are left with the sum

$$n^t \sum_{n^{s \vee t} < k \leq n-n^t} \log \left(1 + \frac{n^s}{k} \right) = n^t \sum_{n^{s \vee t} < k \leq n-n^t} \left(\frac{n^s}{k} + O(n^{2s}/k^2) \right)$$

which evaluates to the main term, as desired. \square

For $\theta \in [\epsilon, \pi - \epsilon]$ we can give a strong approximation to the character ratio $r_{\mathbf{a}(u)}(\theta)$ using the saddle point method. We first record the necessary properties of the function $g_{\mathbf{a}(u)}(z)$.

Lemma 6.2. *Let D be a bounded rectangle contained in the right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$. Let $1/2 \leq u < 1 - \delta$ for some $\delta > 0$, and assume n^u is an integer. Uniformly for $z \in D$ we have for all $m \geq 1$*

$$g_{\mathbf{a}(u)}^{(m)}(z) = g_{\mathbf{0}}^m(z) - n^{2(u-1)} g_{\mathbf{0}}^{m+2}(z) + O_{m,D}(n^{3(u-1)}).$$

In particular, suppose $\theta \in (\epsilon, \pi - \epsilon)$ for some $\epsilon > 0$. Let $\omega' > 0$ solve the saddle point equation $g'_{\mathbf{a}(u)}(\omega') = 0$ and let ω be the usual saddle point $g'_{\mathbf{0}}(\omega) = 0$. We have

$$\omega' = \omega + n^{2(u-1)} \frac{g_{\mathbf{0}}^{(3)}(\omega)}{g_{\mathbf{0}}^{(2)}(\omega)} + O_{\epsilon}(n^{3(u-1)}).$$

Also,

$$g_{\mathbf{a}(u)}(\omega') = g_{\mathbf{0}}(\omega) - n^{2(u-1)} g_{\mathbf{0}}^{(2)}(\omega) + O_{\epsilon}(n^{3(u-1)}), \quad g_{\mathbf{a}(u)}^{(2)}(\omega') = g_{\mathbf{0}}^{(2)}(\omega) + O_{\epsilon}(n^{2(u-1)}).$$

Finally, for all $t > 0$ and all n sufficiently large we have

$$\Re g_{\mathbf{a}(u)}(\omega' + it + i/n) \leq \Re g_{\mathbf{a}(u)}(\omega' + it).$$

Proof. Set $\phi(z) = i[\psi(n + 1/2 - inz) - \psi(-n + 1/2 - inz)]$, and recall that $g'_0(z) = \theta + \phi(z)$. We have

$$g'_{\mathbf{a}(u)}(z) = g'_0(z) + (\phi(z + in^{u-1}) - 2\phi(z) + \phi(z - in^{u-1})).$$

The difference is $n^{2(u-1)}g_0^{(3)}(z) + O(n^{3(u-1)})$. The claims for the other derivatives follow similarly.

The facts regarding ω' and $g_{\mathbf{a}(u)}^{(m)}(\omega')$ may be deduced by standard calculus. The monotonicity property for $g_{\mathbf{a}(u)}$ on $\Re(z) = \omega'$ is proven in the same way as the related claim for g_0 , in Lemma 3.3. \square

Using the last lemma, we can now evaluate $\log |r_{\mathbf{a}(u)}(\theta)|$.

Proposition 6.3. *Uniformly for θ in compact subsets of $(0, \pi)$ and for $1/2 \leq u \leq 2/3$ we have*

$$r_{\mathbf{a}(u)}(\theta) = (1 + O(n^{3u-2})) \exp(-2(\sin \theta/2)^2 n^{2u-1}).$$

Proof. The assumption on θ ensures that $g_0^{(2)}(\omega) = O(1)$. Standard application of the saddle point method gives

$$r_{\mathbf{a}(u)}(\theta) = (1 + O(1/n)) \frac{(2n-1)!}{(2n \sin(\theta/2))^{2n-1}} \frac{e^{ng_{\mathbf{a}(u)}(\omega')}}{\sqrt{2\pi n g_{\mathbf{a}(u)}^{(2)}(\omega')}}.$$

Comparing this to the integral for the trivial representation at its saddle point ω yields

$$\begin{aligned} r_{\mathbf{a}(u)}(\theta) &= \exp(n(g_{\mathbf{a}(u)}(\omega') - g_0(\omega))) (1 + O(n^{2(u-1)})) \\ &= \exp\left(-n^{2u-1}g_0^{(2)}(\omega) + O(n^{3u-2})\right) = \exp\left(-2(\sin \theta/2)^2 n^{2u-1} + O(n^{3u-2})\right). \end{aligned}$$

Here we use $g_0^{(2)}(\omega) \approx 2(\sin \theta/2)^2$, as seen from $\omega \approx \cot \theta/2$, with an error absorbed in the $O(n^{3u-2})$ term. \square

Proof of lower bound in Theorem 1.3. It suffices to prove the following two results separately:

1. For $t = \frac{1}{2\xi(\sigma^2)}n(\log n - c)$, $\|\delta_{Id} \cdot P^t - \nu\|_2 \rightarrow \infty$, $n \rightarrow \infty, c \rightarrow \infty$
2. For $t = \frac{1}{4\sigma^2(q)}n(\log n - 3\log_2 n - c)$, $\|\delta_{Id} \cdot P^t - \nu\|_2 \rightarrow \infty$, $n \rightarrow \infty, c \rightarrow \infty$.

We prove both results by dropping all but one term in the sum

$$\|\delta_{Id} \cdot P^t - \nu\|_2^2 = \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 |\xi(r_{\mathbf{a}})|^{2t}.$$

In the first case, we take the natural representation $\mathbf{a} = (\mathbf{0}, 1)$, with dimension $d_{(\mathbf{0},1)} = 2n + 1$ and character ratio $\xi(r_{(\mathbf{0},1)}) = 1 - \frac{2\xi(\sigma^2)}{n} + O(1/n^2)$, which plainly suffices.

Notice that the second case is contained in the first unless $\sigma^2(q) < \frac{\xi(\sigma^2)}{2} \leq 1/2$, so we assume $\sigma^2(q) < 1/2$ from now on. In this case we consider a representation of the form $\mathbf{a}(u)$, where u is slightly larger than $1/2$. Introduce parameters Δ and β characterized by

$$e^\beta = \frac{1}{2}\Delta \log \Delta = \frac{\log n}{2(\sin q/2)^2}$$

and set $u = \frac{1}{2} + \frac{\beta}{2 \log n}$. Note that our dimension evaluation in Lemma 6.1 gives

$$(1 + O(1/\log n)) \log d_{\mathbf{a}(u)} = \left(\frac{1}{2} - \frac{\log \log n}{2 \log n} \right) e^\beta n \log n.$$

Since q is in the support of ξ , we have

$$\xi(\{\theta : \|\theta\| \in [q, q + \Delta^{-1}]\}) \geq \frac{\delta}{\Delta}$$

for some $\delta > 0$. Proposition 6.3 gives an asymptotic evaluation of $r_{\mathbf{a}(u)}(\theta)$ for $\theta \in [q, \pi - \epsilon]$, so in evaluating the contribution of $\theta \in [\pi - \epsilon, \pi]$ to $\xi(r_{\mathbf{a}(u)})$ we simply put in the bound of Proposition 3.1 for the fixed θ walk,

$$\log |r_{\mathbf{a}(u)}(\theta)| \lesssim -\frac{(2 - \epsilon^2/2) \log d_{\mathbf{a}(u)}}{n(\log n + O(1))}.$$

Putting in the asymptotic of Proposition 6.3 for $r_{\mathbf{a}(u)}(\theta)$ for θ on the small interval $[q, q + \Delta^{-1}]$ and using only that the character ratio is non-negative on the remaining bulk, $\theta \in [q + \Delta^{-1}, \pi - \epsilon]$, we deduce the lower bound

$$\begin{aligned} \xi(r_{\mathbf{a}(u)}) &\geq (1 + O(n^{3u-2})) \int_{\|\theta\| \in [q, q + \Delta^{-1}]} \exp\left(-2(\sin \theta/2)^2 e^\beta\right) d\xi(\theta) \\ &\quad - \int_{|\theta - \pi| < \epsilon} \exp\left(-(1 + O(\epsilon^2))e^\beta\right) d\xi(\theta) \\ &\geq \delta \Delta^{-1} \exp\left(-2(\sin q/2)^2 e^\beta\right) \exp\left(-2\Delta^{-1}e^\beta\right) - \exp\left(-(1 + O(\epsilon^2))e^\beta\right) \end{aligned}$$

Since $\sigma^2(q) < 1/2$ this furnishes an asymptotic of shape

$$\xi(r_{\mathbf{a}(u)}) \gtrsim \frac{\delta}{\Delta^2} \exp\left(-2(\sin q/2)^2 e^\beta\right)$$

if ϵ is chosen sufficiently small [use $\exp(2\Delta^{-1}e^\beta) = \Delta \asymp \frac{\log n}{\log \log n}$ while $e^\beta \asymp \log n$]. Therefore

$$\log \xi(r_{\mathbf{a}(u)}) \geq -2(\sin q/2)^2 e^\beta - 2 \log_2 n - O(1).$$

We deduce that

$$\frac{n(\log n - 3 \log_2 n)}{4(\sin q/2)^2} \log \xi(r_{\mathbf{a}(u)}) + \log d_{\mathbf{a}(u)} \geq O(\log d_{\mathbf{a}(u)}(\log n)^{-1}).$$

The result follows since the error can be absorbed in the constant c . \square

6.2.1 Upper bound for L^2 mixture walk

We now turn to the upper bound in Theorem 1.3. In this section we prove the following variant of the character ratio bound in Proposition 3.1 for mixtures of character ratios.

Proposition 6.4. *Let ξ be a probability measure supported in $(0, \pi)$ and let $q > 0$ and $q' < \pi$ be the smallest and largest points in its support. As usual, let $\sigma(\theta) = \sin(\theta/2)$. Let $\rho_{\mathbf{a}}$ be a non-trivial irreducible representation. There exists a constant $C > 0$ for which the following bound holds.*

$$\log \xi(|r_{\mathbf{a}}|) \leq -\min\left(\frac{2\xi(\sigma^2)}{n(\log n + C)}, \frac{4\sigma^2(q)}{n(\log n + 2 \log_2 n + C)}\right) \log d_{\mathbf{a}}.$$

From this Proposition, the deduction of the upper bound in Theorem 1.3 is the same as for Theorem 1.2.

As in the proof of Proposition 3.1 for fixed angle, the proof of Proposition 6.4 splits according as the representation is small, moderate or large. When the representation is small, in the range $\sum a_j < \frac{400}{\log n}$, one may apply Lemma 5.3 directly to deduce that

$$r_{\mathbf{a}}(\theta) = 1 - \frac{E_1 \sigma^2(\theta)}{n^2} (1 + O(1/\log n)),$$

and that, in this range, $E_1 = O(n^2/(\log^2 n))$. Thus for small representations,

$$\xi(|r_{\mathbf{a}}|) = \xi(r_{\mathbf{a}}) = 1 - \frac{E_1 \xi(\sigma^2)}{n^2} (1 + O(1/\log n))$$

and

$$\log \xi(|r_{\mathbf{a}}|) = -\frac{E_1 \xi(\sigma^2)}{n^2} (1 + O(1/\log n)).$$

The proof that

$$\log \xi(|r_{\mathbf{a}}|) \leq -\frac{2\xi(\sigma^2) \log d_{\mathbf{a}}}{n(\log n + C)}$$

now goes through as before.

Similarly, when the representation is large, in the regime where $a_n \geq \frac{2 \cdot 10^6}{\sigma(q)} n$, arguments similar to our previous ones reduce the problem to the case of small and moderate dimensions. Recall that previously when the representation is large, we either build the character ratio out of a ratio on a smaller group by appending weights, or bound the integral trivially. We leave it to the reader to check that in our bounds from Section 5.3 on large representations, the incremental contributions to the log of the dimension are dominated by the contributions to the character ratio by a factor of at least $\log n$, so that large representations may be ignored.

We now turn to the main case of moderate representations, where there are some new ideas. One idea is that the greatest trade-off between character ratio and dimension growth occurs for a_n of size about $n^{1/2}$. For smaller a_n , the character ratio is sufficiently near 1 that there is no loss in integrating it directly as opposed to its logarithm. For larger a_n the character ratio outstrips the dimension by a larger amount. Another idea, already illustrated in our proof of the L^2 lower bound, is that the dimension is most difficult to control when the indices a_j are shifted in a ‘clump’. We handle this case in Lemma 6.10 below.

Before starting out we make several simplifying reductions. As before, we bound the character ratio $r_{\mathbf{a}}(\theta)$ for $\theta \in [q, q']$ by bounding the associated integral,⁵

$$|r_{\mathbf{a}}(\theta)| \leq \frac{(2n-1)!}{(2n \sin \frac{\theta}{2})^{2n-1}} \frac{1}{2\pi} \int_{|t| \ll_{\xi} \frac{1}{\sqrt{\log n}}} e^{n\Re(g_{\mathbf{a}}(\omega+it))} dt.$$

This we write as

$$\frac{(2n-1)!}{(2n \sin \frac{\theta}{2})^{2n-1}} \frac{1}{2\pi} \int_{|t| \ll_{\xi} \frac{1}{\sqrt{\log n}}} e^{L_{\mathbf{a}}(\theta, t)} e^{n\Re(g_0(\omega+it))} dt \quad (37)$$

⁵Throughout this section we will work with integrals truncated at $|t| \ll_{\xi} \frac{1}{\sqrt{\log n}}$, thus neglecting the tail. The necessary argument to control the error from the tail is the same as in Proof of Proposition 3.1.

where we introduce

$$L_{\mathbf{a}}(\theta, t) = n\Re(g_{\mathbf{a}}(\omega(\theta) + it) - g_0(\omega(\theta) + it)) = \log \left| \prod_{j=1}^n \frac{(\omega(\theta) + it)^2 + \omega_j^2}{(\omega(\theta) + it)^2 + \alpha_j^2} \right|$$

and also $L_{\mathbf{a}}(\theta) = L_{\mathbf{a}}(\theta, 0)$.

We now record several lemmas regarding $L_{\mathbf{a}}(\theta, t)$.

Lemma 6.5. *In the range $\omega = \Theta(1)$, $\alpha_j = O(1)$ and $|t| = o(1)$,*

$$\left| \frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2} \right| = 1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2} [1 + O(t^2)].$$

In particular, if $\rho_{\mathbf{a}}$ is a moderate representation and $\theta \in [q, q']$, $|t| \ll \frac{1}{\log n}$ then

$$L_{\mathbf{a}}(\theta, t) = L_{\mathbf{a}}(\theta)(1 + O(t^2)).$$

Proof. Write

$$\frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2} = 1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2} \left[1 + \left(\frac{\alpha_j^2 + \omega^2}{\alpha_j^2 + (\omega + it)^2} - 1 \right) \right].$$

Since $1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2} = \Omega(1)$, the first statement's proof is completed by noting that $\left(\frac{\alpha_j^2 + \omega^2}{\alpha_j^2 + (\omega + it)^2} - 1 \right)$ has imaginary part that is $O(|t|)$ and real part that is $O(t^2)$.

For the second statement, note that $\rho_{\mathbf{a}}$ moderate implies $\alpha_j = O(1)$, while $\theta \in [q, q']$ implies $\omega = \theta(1)$. The first statement then applies, and shows that

$$\log \left| \frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2} \right| = (1 + O(t^2)) \log \frac{\omega^2 + \omega_j^2}{\omega^2 + \alpha_j^2}.$$

The second claim follows on summing over j . □

Lemma 6.6. *Let $\rho_{\mathbf{a}}$ be a moderate representation and let $\theta \in [q, q']$. Then $L_{\mathbf{a}}(\theta) \asymp L_{\mathbf{a}}(q)$. Also, $-L_{\mathbf{a}}(q) \gg \frac{1}{\log n}$ for all moderate \mathbf{a} .*

Proof. Since $\omega = \Theta(1)$ and $\alpha_j = O(1)$ for all j the first statement follows immediately from the definition,

$$L_{\mathbf{a}}(\theta) = \sum \log \left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega^2 + \alpha_j^2} \right).$$

For the second statement, recall that \mathbf{a} moderate guarantees $\sum_j a_j \gg \frac{n}{\log n}$ and $a_n \ll n$. Since the a_j are weakly increasing, $\sum_{j>n/2} a_j \gg \frac{n}{\log n}$. Thus

$$-L_{\mathbf{a}}(q) \gg - \sum_{j>n/2} \log \left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega^2 + \alpha_j^2} \right) \gg \sum_{j>n/2} (\alpha_j - \omega_j) \gg \frac{1}{\log n}$$

since $1 \ll \omega_j \leq \alpha_j \ll 1$. □

Lemma 6.7. *There is a constant $c > 0$ such that for all moderate representations $\rho_{\mathbf{a}}$ we have*

$$\log \xi(|r_{\mathbf{a}}|) \leq cL_{\mathbf{a}}(q).$$

Furthermore, there is a $c'(\xi) > 0$ such that if $-L_{\mathbf{a}}(q) < c'n$ then

$$\log \xi(|r_{\mathbf{a}}|) \leq (1 + O(\log n/n)) \log \xi(e^{L_{\mathbf{a}}(\theta)}).$$

Proof. By modeling the proof of Proposition 3.1 for moderate representations (see e.g. (33)) we have

$$\xi(|r_{\mathbf{a}}|) \leq (1 + O(1/n)) \sup_{\theta \in [q, q']} \sup_{|t| \leq \frac{1}{\sqrt{\log n}}} e^{-L_{\mathbf{a}}(\theta, t)}.$$

The first statement follows since $L_{\mathbf{a}}(\theta, t) \sim L_{\mathbf{a}}(\theta) \asymp L_{\mathbf{a}}(q)$ [the error term of size $O(1/n)$ is negligible].

To prove the second statement, use (37) to write

$$\begin{aligned} |r_{\mathbf{a}}(\theta)| &\leq e^{L_{\mathbf{a}}(\theta)} \frac{(2n-1)! e^{ng_0(\omega)}}{(2n \sin(\frac{\theta}{2}))^{2n-1}} \\ &\quad \times \frac{1}{2\pi} \int_{|t| \leq \frac{1}{\sqrt{\log n}}} \exp \left(-\frac{ng_0^{(2)}(\omega)t^2}{2} + O(t^2 L_{\mathbf{a}}(\theta)) + O(nt^4) \right) dt. \end{aligned}$$

For $L_{\mathbf{a}}(\theta) < c'n$ for a sufficiently small constant c'' , the entire expression evaluates to (see (9))

$$e^{L_{\mathbf{a}}(\theta)} (1 + O(1/n) + O(L_{\mathbf{a}}(\theta)/n)).$$

Thus

$$\xi(|r_{\mathbf{a}}|) \leq \xi(e^{L_{\mathbf{a}}}) (1 + O(1/n) + O(L_{\mathbf{a}}(q)/n)).$$

The final claim follows since $\log \xi(e^{L_{\mathbf{a}}}) \asymp L_{\mathbf{a}}(q)$ and $-L_{\mathbf{a}}(q) \gg 1/\log n$. \square

The above lemmas effectively reduce our problem to that of comparing $L_{\mathbf{a}}(\theta)$ to the dimension $\log d_{\mathbf{a}}$. Recall that earlier we introduced the dimension increment

$$d_{\mathbf{a}}(k) = \frac{\tilde{a}_k}{2k-1} \prod_{j < k} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k-1/2)^2 - (j-1/2)^2}$$

and that in Lemma 4.1 we proved the bound

$$\log d_{\mathbf{a}}(k) \leq a_k \log \left(1 + \frac{2k-1}{a_k} \right) + (2k-1) \log \left(1 + \frac{a_k}{2k-1} \right) + O(1). \quad (38)$$

Similarly, now we introduce the character ratio increment (recall $\alpha_j = (a_j + j - 1/2)/n$)

$$L_{\mathbf{a}}(k, \theta) = \log \left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega(\theta)^2 + \alpha_j^2} \right).$$

We now prove several lemmas about these increments.

Lemma 6.8. *Let $\rho_{\mathbf{a}}$ be a moderate representation. For all $j \gg n$ we have*

$$-L_{\mathbf{a}}(j, \theta) \gg_{\xi} \frac{a_j}{n}.$$

If $j > n - Cn/\log n$ for some constant C and $a_j < n/\log n$ then

$$-L_{\mathbf{a}}(j, \theta) = \frac{2\sigma^2(\theta)a_j}{n} (1 + O(1/\log n)). \quad (39)$$

Proof. For both claims, write $\alpha_j^2 - \omega_j^2 = \frac{a_j}{n} \cdot \frac{a_j + 2j - 1}{n}$. The first claim is straightforward. For the second, observe that $\frac{a_j + 2j - 1}{n} = 2 + O(1/\log n)$ and $\frac{1}{\omega_j^2 + \alpha_j^2} = \sigma^2(1 + O(1/\log n))$. \square

The next lemma allows us to prove the type of estimate that we want for Proposition 6.4 for a collection of increments whose sum is not too large.

Lemma 6.9. *Let $G \subset \mathbb{Z} \cap [n - Cn/\log n, n]$, C a constant, be a subset of indices such that $\sum_{j \in G} a_j < \frac{n}{\log n}$. Then for a sufficiently large constant C' ,*

$$\log \int_q^{q'} \exp \left(\sum_{j \in G} L_{\mathbf{a}}(j, \theta) \right) d\xi(\theta) \leq -\frac{2\xi(\sigma^2) \sum_{j \in G} \log d_{\mathbf{a}}(j)}{n(\log n + C')}.$$

Proof. Since $a_j < n/\log n$ for all $j \in G$, we may substitute the asymptotic (39) into the LHS and integrate. Since the argument of the exponential is $O(1/\log n)$, the LHS becomes

$$(1 + O(1/\log n)) \frac{-2\xi(\sigma^2)}{n} \sum_{j \in G} a_j.$$

The result now follows on noting that, in this range, $\log d_{\mathbf{a}}(j) \leq a_j(\log n + O(1))$. \square

Lemma 6.10. *Let B be the collection of indices*

$$B = \{j \in [n - Cn/\log n, n] : a_j \in ((1 - \delta)S, S]\}$$

where $\delta = \frac{1}{\log n}$, C is a constant, and $1 \leq S \leq \frac{n^{1/2}}{1 + \log n}$. Suppose that $|B| \geq \frac{n^{1/2}}{\log n}$ (in addition to $|B| \leq \frac{Cn}{\log n}$). Then

$$\log d_{\mathbf{a}}(B) \stackrel{\text{def}}{=} \sum_{j \in B} \log d_{\mathbf{a}}(j) \leq \left(1 - \frac{\log |B|}{\log n}\right) |B| S (\log n + O(1)).$$

In particular, for each $\theta \in [q, q']$ we have

$$\sum_{j \in B} L_{\mathbf{a}}(j, \theta) \leq \frac{-4\sigma^2(q) \log d_{\mathbf{a}}(B)}{n(\log n + 2\log_2 n + O(1))}.$$

Proof. To deduce the final statement from the remainder of the lemma, observe that for each $j \in B$, Lemma 6.8 gives

$$-L_{\mathbf{a}}(j, \theta) = \frac{2\sigma^2(\theta)S}{n} (1 + O(1/\log n))$$

so that

$$\sum_{j \in B} L_a(j, \theta) \leq -\frac{2\sigma^2(q)|B|S}{n}(1 + O(1/\log n)),$$

while the first part of the present lemma gives that

$$d_{\mathbf{a}}(B) \leq \left(\frac{1}{2} + \frac{\log_2 n}{\log n}\right) |B|S(\log n + O(1)).$$

Turning to the main statement of the lemma, let m be the last index in B , and, with an eye toward applying the block dimension bound in Lemma 6.1, set $T = |B|$. Obviously those indices greater than m will not affect $d_{\mathbf{a}}(B)$, so henceforth we will write \mathbf{a} in place of $\mathbf{a}(m)$, the weight truncated at m corresponding to a representation on $SO(2m+1)$.

Factor $d_{\mathbf{a}}(B)$ as $d_{\mathbf{a}}(B) = E_{\mathbf{a}}(B) \cdot I_{\mathbf{a}}(B)$ where $E_{\mathbf{a}}(B)$ corresponds to factors pairing indices in B with those outside,

$$E_{\mathbf{a}}(B) = \prod_{j \in B} \frac{\tilde{a}_j}{j-1/2} \prod_{i \leq m-T} \frac{\tilde{a}_j^2 - \tilde{a}_i^2}{(j-1/2)^2 - (i-1/2)^2},$$

and where $I_{\mathbf{a}}(B)$ pairs indices both inside B ,

$$I_{\mathbf{a}}(B) = \prod_{j \in B} \prod_{m-T \leq i < j} \frac{\tilde{a}_j^2 - \tilde{a}_i^2}{(j-1/2)^2 - (i-1/2)^2}.$$

Set $s = \frac{\log S}{\log m}$, $t = \frac{\log T}{\log m}$ and write as before $\mathbf{a}(S, T) = (\mathbf{0}_{m-m^t}, (m^s)_{m^t})$. Then we may bound $E_{\mathbf{a}}(B) \leq d_{\mathbf{a}(s,t)}$ so that Lemma 6.1 yields the bound

$$\log E_{\mathbf{a}}(B) \leq (1 - s \vee t)ST(\log n + O(1)).$$

This is of the right size, so we now work to show that $\log I_{\mathbf{a}}(B)$ is of lower order by a factor of $\log n$.

Write $I_{\mathbf{a}}(B) = I_{\mathbf{a}}(B)^+ I_{\mathbf{a}}(B)^-$, where

$$I_{\mathbf{a}}(B)^+ = \prod_{j \in B} \prod_{m-T < i < j} \frac{a_i + a_j + i + j - 1}{i + j - 1}, \quad I_{\mathbf{a}}(B)^- = \prod_{j \in B} \prod_{m-T < i < j} \frac{a_j - a_i + j - i}{j - i}$$

Then we have the bound

$$\log I_{\mathbf{a}}(B)^+ \leq \sum_{j \in B} \sum_{m-T < i < j} \log \left(1 + \frac{S}{n(1 - O(1/\log n))} \right) \leq (1 + O(1/\log n)) \frac{ST^2}{2n},$$

which more than suffices, since $T \leq Cn/\log n$. To bound $I_{\mathbf{a}}(B)^-$ consider the representation $\rho_{\mathbf{b}}$ on $SO(2T+1)$ with $b_i = a_{m-T+i} - a_{m-T} \leq \delta S$. Then $I_{\mathbf{a}}(B)^- = d_{\mathbf{b}}^- \leq d_{\mathbf{b}}$. Since $S \leq T$, the bound (38) gives [use $\log(1+x) \leq x$]

$$\log d_{\mathbf{b}} \leq \sum_{i=1}^T b_i(\log T + O(1)) \leq \delta ST(\log n + O(1)).$$

Since $\delta = 1/\log n$ this completes the proof. \square

Lemma 6.11. *Let $\rho_{\mathbf{a}}$ be a moderate representation with $a_j = 0$ for $j < k$. Let \mathbf{a}' denote the string such that $a'(i) = a(i)$ for $i < k$ and $a'(i) = a(i) + 1$ for $i \geq k$. For some constants $C, C' > 0$*

$$\log d_{\mathbf{a}'} \leq \log d_{\mathbf{a}} + Cn$$

while

$$L_{\mathbf{a}'} \leq L_{\mathbf{a}} - C' \left(\frac{n-k}{n} \right)$$

Proof. The dimension claim follows from Lemmas 4.2 and 4.3. The character ratio claim follows on checking that for each $i \geq k$ we have $\log \left(1 - \frac{\alpha_i^2 - \omega_i^2}{\omega_i^2 + \alpha_i^2} \right)$ decreases by $\Omega(1/n)$ after the shift. \square

Proof of Theorem 1.3 upper bound, moderate representations. We begin with a few observations. We assume that $\rho_{\mathbf{a}}$ is moderate, with $a_n \leq \frac{2 \cdot 10^6}{\sigma(q)} n$. Therefore, by Lemma 5.4 of Section 5.2, $\log d_{\mathbf{a}} \ll n^2$. Therefore, we may restrict attention to character ratios for which $-\log \xi(|r_{\mathbf{a}}|) \ll \frac{n}{\log n}$. By Lemmas 6.6 and 6.7 of this section, this means that we may assume $-L_{\mathbf{a}}(q) \ll \frac{n}{\log n}$, so that, to within negligible error, we may replace

$$\xi(|r_{\mathbf{a}}|) \quad \leftrightarrow \quad \xi(e^{L_{\mathbf{a}}(\theta)}).$$

Next observe that for a sufficiently large constant c , we may assume $a_i = 0$ for all $i < n - \frac{cn}{\log n}$. This is because any string with non-zero entries in this region may be obtained from one with all zeros by making shifts of the type described in Lemma 6.11, and if c is sufficiently large, these shifts improve the bound in Proposition 6.4.

We next dispatch with any indices for which $a_j > \frac{n^{1/2}}{\log n}$. Recall that $a_j \ll n$ so that (38) gives

$$\log d_{\mathbf{a}}(j) \leq a_j(\log n - \log a_j + O(1)).$$

If $a_j \geq \frac{n}{(\log n)^2}$ then the first part of Lemma 6.8 gives $L_{\mathbf{a}}(j, \theta) \gg \frac{a_j}{n}$. Thus

$$-\frac{n \log n}{4\sigma^2(q)} L_{\mathbf{a}}(j, \theta) \gg \frac{\log n}{\log_2 n} \log d_{\mathbf{a}}(j).$$

If $a_j \leq \frac{n}{(\log n)^2}$ then the second part of Lemma 6.8 guarantees that, for a sufficiently large constant C ,

$$\frac{n(\log n + 2 \log_2 n + C)}{4\sigma^2(q)} L_{\mathbf{a}}(j, \theta) + \log d_{\mathbf{a}}(j) \leq 0,$$

but now the factor of $\log_2 n$ is needed.

Now we treat the indices for which $a_j < n^{1/2}/\log n$ by splitting them into bins. Let $\delta = 1/\log n$ and let

$$I_k = \left[\frac{n^{1/2}}{\log n} (1 - \delta)^k, \frac{n^{1/2}}{\log n} (1 - \delta)^{k-1} \right], \quad 1 \leq k \leq K = \frac{\log \frac{n^{1/2}}{\log n}}{-\log(1 - \delta)} \sim n^{1/2}.$$

Say an interval I_k is ‘big’ if there are at least $n^{1/2}/\log n$ indices j for which $a_j \in I_k$. By Lemma 6.1, if I_k is big then we have, for a sufficiently large C , for all $\theta \in [q, q']$,

$$\sum_{j: a_j \in I_k} L_{\mathbf{a}}(j, \theta) \leq -\frac{4\sigma^2(q)}{n(\log n + 2 \log_2 n + C)} \sum_{j: a_j \in I_k} \log d_{\mathbf{a}}(j).$$

Let \mathcal{B} be the collection of all indices j belonging to a big interval, together with all j for which $a_j > n^{1/2}/\log n$. Let $m \geq n - cn/\log n$ be the least index for which $a_m \neq 0$, and set $\mathcal{S} = [m, n] \setminus \mathcal{B}$. Observe that

$$L_{\mathbf{a}}(\theta) = \sum_{j \in \mathcal{B}} L_{\mathbf{a}}(j, \theta) + \sum_{j \in \mathcal{S}} L_{\mathbf{a}}(j, \theta) = L_{\mathbf{a}, \mathcal{B}}(\theta) + L_{\mathbf{a}, \mathcal{S}}(\theta),$$

and

$$\log d_{\mathbf{a}} = \sum_{j \in \mathcal{B}} d_{\mathbf{a}}(j) + \sum_{j \in \mathcal{S}} d_{\mathbf{a}}(j) = \log d_{\mathbf{a}, \mathcal{B}} + \log d_{\mathbf{a}, \mathcal{S}}.$$

Since each $j \in \mathcal{S}$ has a_j in an interval that contains at most $n^{1/2}/\log n$ other indices, we may bound

$$\sum_{j \in \mathcal{S}} a_j \leq \frac{n^{1/2}}{\log n} \sum_{k=0}^{K-1} (1-\delta)^k \frac{n^{1/2}}{\log n} \leq \frac{n}{\log n},$$

and, therefore, by Lemma 6.9,

$$\log \int_q^{q'} e^{L_{\mathbf{a}, \mathcal{S}}(\theta)} d\xi(\theta) \leq -\frac{2\xi(\sigma^2) \log d_{\mathbf{a}, \mathcal{S}}}{n(\log n + C)}.$$

Now we bound

$$\begin{aligned} \log \xi(|r_{\mathbf{a}}|) &\leq \log \int_q^{q'} e^{L_{\mathbf{a}}(\theta)} d\xi(\theta) \\ &\leq \sup_{\theta \in [q, q']} L_{\mathbf{a}, \mathcal{B}}(\theta) + \log \int_q^{q'} e^{L_{\mathbf{a}, \mathcal{S}}(\theta)} d\xi(\theta) \\ &\leq -\frac{4\sigma^2(q) \log d_{\mathbf{a}, \mathcal{B}}}{n(\log n + 2\log_2 n + C)} - \frac{2\xi(\sigma^2) \log d_{\mathbf{a}, \mathcal{S}}}{n(\log n + C)}, \end{aligned}$$

completing the proof. \square

Remark 5. From the proof of Theorem 1.3, we also get cutoff for arbitrary L^2 distance gap. Let $T_{2,k}(n) := \inf\{t > 0 : \|P^t - \nu_n\|_2 < k\}$. Then there exists $c_k < \infty$ such that

$$\frac{n(\log n - 3\log_2 n - c_k)}{4\sigma^2(q)} < T_{2,k}(n) < \frac{n(\log n + 2\log_2 n + c_k)}{2\xi(\sigma^2)}.$$

In particular, $\lim_k \lim_n T_{2,k}(n)/(n \log n) = \frac{1}{4\sigma^2(q) \wedge 2\xi(\sigma^2)}$.

7 L^∞ mixing time

In this section we prove Corollary 1.2. First we define, for two Borel probability measures on a compact metric space (Ω, ρ) , the L^∞ distance of μ and ν w.r.t. ν , by

$$\|\mu - \nu\|_\infty := \|\mu - \nu\|_{\infty, \nu} := \sup_{f: \nu(f) \leq 1} |\mu(f) - \nu(f)|.$$

If μ is not absolutely continuous with respect to ν then the L^∞ norm is ∞ , since if $U = \{x \in \Omega : \frac{d\mu}{d\nu}(x) = \infty\}$ then for any Borel function f supported on U , $\nu(f+1) = 1$. Conversely, if μ has a density with respect to ν , then we have the alternative definition:

$$\|\mu - \nu\|_\infty = \text{ess sup}_{x \in \Omega} \left| \frac{d\mu}{d\nu}(x) - 1 \right|.$$

The equivalence of the two definitions is trivial in the discrete case. For compact metric space, use the fact that $\frac{d\mu}{d\nu}(x) = \lim_{r \rightarrow 0^+} \frac{\mu(B_r(x))}{\nu(B_r(x))}$, where $B_r(x)$ is metric ball of radius r around x .

Now specializing to the running distributions of a Markov chain, the first definition further implies that for any starting state x , the L^∞ distance $d_\infty(t) := \|P_x^t - \nu\|_\infty$ is monotone non-increasing, since

$$\begin{aligned} \|P_x^{t+1} - \nu\|_\infty &= \|P_x^t P - \nu P\|_\infty = \sup_{f: \nu(f) \leq 1} |P_x^t(P(f)) - \nu(P(f))| \\ &\leq \sup_{g: \nu(g) \leq 1} |P_x^t(g) - \nu(g)| = \|P_x^t - \nu\|_\infty, \end{aligned}$$

using $\nu(f) \leq 1$ implies $\nu(P(f)) \leq 1$, by Markov property.

Now recall the Fourier inversion formula for compact Lie groups: for any $f \in L^2(SO(N))$,

$$f(x) = \sum_{\rho \in \widehat{SO(N)}} d_\rho \text{Tr}[\hat{f}(\rho) \rho(x)^*].$$

Let $f_t = \frac{d\mu_t}{d\nu} - 1$, where $\mu_t = \delta_{\text{Id}} P^t$. Since we are interested in the situation where $\text{ess sup}_{x \in \Omega} \frac{d\mu_t}{d\nu}(x) < \infty$, certainly we may assume $f_t \in L^2(SO(N))$. Since $\mu_t = \mu^{*t}$ is a convolution product and invariant under conjugation, $\hat{f}_t(\rho_{\mathbf{a}}) = \xi(r_{\mathbf{a}}(\theta))^t I_{d_{\mathbf{a}}}$, that is, a constant times the identity matrix. In particular, $\hat{f}_t(\rho_0) = 1$. Therefore

$$f_t(x) = \sum_{\mathbf{a} \neq 0} d_{\mathbf{a}}^2 r_{\mathbf{a}}^t \chi_{\mathbf{a}}(x); \quad \chi_{\mathbf{a}}(x) = \frac{1}{d_{\mathbf{a}}} \text{Tr} \rho_{\mathbf{a}}(x).$$

Thus when $t = 2s$ is even, the maximum of $|f_t(x)|$ occurs at $x \in \{\pm \text{Id}\}$ ($r_{\mathbf{a}} := \frac{1}{d_{\mathbf{a}}} \hat{\mu}(\rho)$ can be either positive or negative). But at $x = \text{Id}$ say, we have $f_t(\text{Id}) = \sum_{\mathbf{a}}^* d_{\mathbf{a}}^2 r_{\mathbf{a}}^t$, which coincides with $\|P_{\text{Id}}^s - \nu_n\|_2$. Thus $\|P_{\text{Id}}^t - \nu_n\|_\infty = \|P_{\text{Id}}^s - \nu_n\|_2$ and the Corollary follows from monotonicity of the L^∞ distance.

8 Open problems

We have seen the power of the contour representation formula (4) in proving sharp estimates for the size of the character ratio, which is instrumental in studying the convergence rate of the generalized Rosenthal walk under various norms. Several questions however still remain:

- What is the optimal cutoff window of L^2 and L^∞ convergence? We show only show that it is of order $O(n \log_2 n)$, but our proof accommodates a general measure. It seems likely that for a specific measure $O(n)$ is the right answer.

- Theorem 1.3 artificially assumes that the generating measure μ be supported away from $\pi \in \mathbb{T}^1$. The assumption is necessary in our proof because the saddle point method breaks down for θ close to π . Incidentally, the perturbative approach based on Rosenthal's $\theta = \pi$ upper bound result is not applicable here since it does not allow the separation of small and medium weights in our analysis.
- For a fixed n it would be nice to determine the first step at which the L^2 distance from uniform becomes finite. Carefully tracing our treatment of large representations shows that this happens in time $O(n)$, while a time $\geq n$ is necessary, because prior to this point the measure is supported on a lower dimensional submanifold.
- In Appendix A we prove a generalization of the the contour formula (4) to the case where the generating measure of the random walk is supported on bigger conjugacy classes of $SO(N)$, such as $SO(2k) \subset SO(N)$; this is derived in Appendix A below. It remains to be seen whether multidimensional analogues of our arguments yield the corresponding mixing time analysis for the related random walks. A reasonable lower bound and candidate mixing time is available in this case from the second moment method applied to the natural representation.
- In [2] it is proved that under very general condition on finite state space Markov chains, ℓ^p -cutoff implies ℓ^q cutoff for $q > p$. Does the same hold for our continuous state space walk?

A Contour formula for characters of $SO(2n+1)$

We have hitherto used an integral expression for the character ratios of the orthogonal group evaluated at a rotation in a single plane. We now generalize this expression to give a k -fold integral formula for the character ratios evaluated at a generic element of $SO(2k) \subset SO(2n+1)$.

Let $\rho_{\mathbf{a}}$, $\mathbf{a} = a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, be an irreducible representation of $SO(2n+1)$ corresponding to highest weight $\sum a_i L_i$ (the L_i are roots), and let $\theta = (\theta_1, \dots, \theta_k) \in (\mathbb{R}/2\pi\mathbb{Z})^k$ be an element of a torus contained in $SO(2k) \subset SO(2n+1)$. Recall that in the case $k = 1$, we derived from Rosenthal's work [17] the formula ($\tilde{a}_j = a_j + n - j + 1/2$, $\tilde{0}_j = n - j + 1/2$)⁶

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n \sin \theta/2)^{2n-1}} \oint M_{\mathbf{a};\theta}(z) dz; \quad M_{\mathbf{a};\theta}(z) = \frac{\sin \theta z}{\prod_{j=1}^n (\tilde{a}_j^2 - z^2)}.$$

The general formula is given by

$$r_{\mathbf{a}}(\theta_1, \dots, \theta_k) = L(\theta_1, \dots, \theta_k) \oint \cdots \oint_{\mathcal{C}_1 \times \dots \times \mathcal{C}_k} M_{\mathbf{a};\theta_1, \dots, \theta_k}(z_1, \dots, z_k) dz_1, \dots, z_k,$$

where the contours \mathcal{C}_i are chosen to have winding number 1 about each of the poles $\pm \tilde{a}_i$ of the integrand

$$M_{\mathbf{a};\theta_1, \dots, \theta_k}(z_1, \dots, z_k) = \prod_{j=1}^k M_{\mathbf{a};\theta_j}(z_j) \times \prod_{1 \leq q \neq r \leq k} (z_q^2 - z_r^2).$$

⁶By the notation \oint we mean integration around a differentiable loop with winding number 1 about all points in its interior. A factor of $\frac{1}{2\pi i}$ is included.

The leading term $L(\theta) = M_0(\theta)^{-1}$ is independent of the representation \mathbf{a} and given by

$$L(\theta_1, \dots, \theta_k) = \frac{(2n-1)!(2n-3)! \dots (2n-2k+1)!}{2^{2nk-k^2} \prod_{j=1}^k \sin(\theta_j)^{2n-2j+1} \prod_{1 \leq i < j \leq k} (\sin(\theta_i/2)^2 - \sin(\theta_j/2)^2)}.$$

The character $\chi_{\mathbf{a}}(\theta)$ is given by the following determinantal formula ([7], p. 408, (24.28)) ⁷

$$\chi_{\mathbf{a}}(\theta) = \frac{\begin{vmatrix} s(\tilde{a}_1\theta_1) & s(\tilde{a}_1\theta_2) & \cdot & s(\tilde{a}_1\theta_n) \\ s(\tilde{a}_2\theta_1) & s(\tilde{a}_2\theta_2) & \cdot & s(\tilde{a}_2\theta_n) \\ \cdot & \cdot & \cdot & \cdot \\ s(\tilde{a}_n\theta_1) & s(\tilde{a}_n\theta_2) & \cdot & s(\tilde{a}_n\theta_n) \end{vmatrix}}{\begin{vmatrix} s(\tilde{0}_1\theta_1) & s(\tilde{0}_1\theta_2) & \cdot & s(\tilde{0}_1\theta_n) \\ s(\tilde{0}_2\theta_1) & s(\tilde{0}_2\theta_2) & \cdot & s(\tilde{0}_2\theta_n) \\ \cdot & \cdot & \cdot & \cdot \\ s(\tilde{0}_n\theta_1) & s(\tilde{0}_n\theta_2) & \cdot & s(\tilde{0}_n\theta_n) \end{vmatrix}}$$

Since the dimension is equal to the character value at $\theta = 0$, we obtain the character ratio

$$r_{\mathbf{a}}(\theta_1, \dots, \theta_k) = \frac{\begin{vmatrix} s(\tilde{a}_1\theta_1) & \cdot & s(\tilde{a}_1\theta_k) & s(\tilde{a}_1\epsilon_{k+1}) & \cdot & s(\tilde{a}_1\epsilon_n) \\ s(\tilde{a}_2\theta_1) & \cdot & s(\tilde{a}_2\theta_k) & s(\tilde{a}_2\epsilon_{k+1}) & \cdot & s(\tilde{a}_2\epsilon_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s(\tilde{a}_n\theta_1) & \cdot & s(\tilde{a}_n\theta_k) & s(\tilde{a}_n\epsilon_{k+1}) & \cdot & s(\tilde{a}_n\epsilon_n) \end{vmatrix}}{\begin{vmatrix} s(\tilde{a}_1\epsilon_1) & s(\tilde{a}_1\epsilon_2) & \cdot & s(\tilde{a}_1\epsilon_n) \\ s(\tilde{a}_2\epsilon_1) & s(\tilde{a}_2\epsilon_2) & \cdot & s(\tilde{a}_2\epsilon_n) \\ \cdot & \cdot & \cdot & \cdot \\ s(\tilde{a}_n\epsilon_1) & s(\tilde{a}_n\epsilon_2) & \cdot & s(\tilde{a}_n\epsilon_n) \end{vmatrix}} \times \frac{\begin{vmatrix} s(\tilde{0}_1\epsilon_1) & s(\tilde{0}_1\epsilon_2) & \cdot & s(\tilde{0}_1\epsilon_n) \\ s(\tilde{0}_2\epsilon_1) & s(\tilde{0}_2\epsilon_2) & \cdot & s(\tilde{0}_2\epsilon_n) \\ \cdot & \cdot & \cdot & \cdot \\ s(\tilde{0}_n\epsilon_1) & s(\tilde{0}_n\epsilon_2) & \cdot & s(\tilde{0}_n\epsilon_n) \end{vmatrix}}{\begin{vmatrix} s(\tilde{0}_1\theta_1) & \cdot & s(\tilde{0}_1\theta_k) & s(\tilde{0}_1\epsilon_{k+1}) & \cdot & s(\tilde{0}_1\epsilon_n) \\ s(\tilde{0}_2\theta_1) & \cdot & s(\tilde{0}_2\theta_k) & s(\tilde{0}_2\epsilon_{k+1}) & \cdot & s(\tilde{0}_2\epsilon_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s(\tilde{0}_n\theta_1) & \cdot & s(\tilde{0}_n\theta_k) & s(\tilde{0}_n\epsilon_{k+1}) & \cdot & s(\tilde{0}_n\epsilon_n) \end{vmatrix}},$$

where $\epsilon_1, \dots, \epsilon_n$ represent infinitesimals tending to 0.

Let $\epsilon_1 \succ \epsilon_2 \succ \dots \succ \epsilon_n$ where $a \succ b$ means b is eventually smaller than any fixed power of a as a and b tend to 0. Then expanding \sin in its power series about 0 and taking the limit we may write $r_{\mathbf{a}} = R_{\mathbf{a}}/R_0$ where

$$R_{\mathbf{a}} = \frac{\begin{vmatrix} s(\tilde{a}_1\theta_1)/\tilde{a}_1 & \cdot & s(\tilde{a}_1\theta_k)/\tilde{a}_1 & (-\tilde{a}_1^2)^{n-k-1} & (-\tilde{a}_1^2)^{n-k-2} & \cdot & 1 \\ s(\tilde{a}_2\theta_1)/\tilde{a}_2 & \cdot & s(\tilde{a}_2\theta_k)/\tilde{a}_2 & (-\tilde{a}_2^2)^{n-k-1} & (-\tilde{a}_2^2)^{n-k-2} & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s(\tilde{a}_n\theta_1)/\tilde{a}_n & \cdot & s(\tilde{a}_n\theta_k)/\tilde{a}_n & (-\tilde{a}_n^2)^{n-k-1} & (-\tilde{a}_n^2)^{n-k-2} & \cdot & 1 \end{vmatrix}}{\begin{vmatrix} (-\tilde{a}_1^2)^{n-1} & (-\tilde{a}_1^2)^{n-2} & \cdot & 1 \\ (-\tilde{a}_2^2)^{n-1} & (-\tilde{a}_2^2)^{n-2} & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ (-\tilde{a}_n^2)^{n-1} & (-\tilde{a}_n^2)^{n-2} & \cdot & 1 \end{vmatrix}}$$

⁷Henceforth we write only s for \sin .

In the case $k = 1$, expand the first column and use the Vandermonde formula to find

$$R_{\mathbf{a}}(\theta_1) = \sum_{i=1}^n \frac{\sin(\theta_1 \tilde{a}_i)}{\tilde{a}_i \prod_{j \neq i} (\tilde{a}_j^2 - \tilde{a}_i^2)} = \oint M_{\mathbf{a}, \theta_1}(z) dz.$$

In general, expanding by the first k columns we obtain

$$R_{\mathbf{a}}(\theta_1, \dots, \theta_k) = \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ \text{distinct}}} \frac{\prod_{j=1}^k \sin(\theta_j \tilde{a}_{i_j}) \prod_{1 \leq q \neq r \leq k} (\tilde{a}_{i_q}^2 - \tilde{a}_{i_r}^2)}{\prod_{j=1}^k \left(\tilde{a}_{i_j} \prod_{\ell \neq i_j} (\tilde{a}_\ell^2 - \tilde{a}_{i_j}^2) \right)} = \oint \cdots \oint M_{\mathbf{a}, \theta}(\mathbf{z}) d\mathbf{z},$$

since the factor $\prod_{1 \leq i \neq j \leq k} (z_i^2 - z_j^2)$ in $M_{\mathbf{a}, \theta}(\mathbf{z})$ exactly produces the required factor of

$$\prod_{1 \leq q \neq r \leq k} (\tilde{a}_{i_q}^2 - \tilde{a}_{i_r}^2)$$

at each set of poles.

Backtracking a bit, we determine the constant term by

$$R_0 = \frac{\epsilon_1^{2n-1} \epsilon_2^{2n-3} \dots \epsilon_k^{2n-2k+1}}{(2n-1)!(2n-3)! \dots (2n-2k+1)!} \frac{\det \left[s(\tilde{\theta}_i \theta_j) \Big|_{j \leq k} s(\tilde{\theta}_i \epsilon_j) \right]}{\det [s(\tilde{\theta}_i \epsilon_j)]}$$

Since $s((j-1/2)\theta)$ is a polynomial in $s(\theta/2)$ with highest order term $2^{j-1}s(\theta/2)^{2j-1}$, performing row reductions we obtain

$$R_0 = \frac{\epsilon_1^{2n-1} \epsilon_2^{2n-3} \dots \epsilon_k^{2n-2k+1}}{(2n-1)!(2n-3)! \dots (2n-2k+1)!} \frac{\det \left[s(\theta_j/2)^{2n-2i+1} \Big|_{j \leq k} s(\epsilon_j/2)^{2n-2i+1} \right]}{\det [s(\epsilon_j/2)^{2n-2i+1}]}$$

so that, applying the Vandermonde and keeping in mind the relative size of the infinitesimals, we arrive at

$$R_0 = \frac{\prod_{j=1}^k [2^{2n-2j+1} s(\theta_j)^{2n-2j+1}] \prod_{1 \leq i < j \leq k} (s(\theta_i/2)^2 - s(\theta_j/2)^2)}{(2n-1)!(2n-3)! \dots (2n-2k+1)!}$$

as wanted.

B Contour formula for characters of S_n

For an application of the results in this appendix, see [9].

Let $\lambda, \rho \vdash n$ be two partitions of n with λ indexing an f^λ dim. irrep. of S_n , and ρ indexing a conjugacy class. The following expresses the character ratio $\frac{\chi_\rho^\lambda}{f^\lambda}$ as a (multiple) contour integral. The formula in the case where $\rho = (r, 1^{n-r})$ is a cycle is a famous formula of Frobenius, see [13], p. 118.

Write $\mu = \lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n)$. The situation is simplest when $\rho = (r, 1^{n-r})$ has a single non-trivial cycle, in which case the formula is a single contour integral. In this case,⁸

$$\frac{\chi_\rho^\lambda}{f^\lambda} = \frac{(n-r)!}{n!} \oint M_{\mu; r}(z) dz; \quad M_{\mu; r}(z) = -\frac{z^r}{r} \prod_{i=1}^n \frac{z - \mu_i - r}{z - \mu_i}, \quad (40)$$

⁸ $x^{\underline{a}} = x(x-1)\dots(x-a+1)$.

where the contour is taken to be any one containing the residues μ_i . In general, if $\rho = (r_1, r_2, \dots, r_k, 1^{n-r})$ has k non-trivial cycles, then the formula is a contour integration in k variables given by ($r = \sum r_i$)

$$\frac{\chi_\rho^\lambda}{f^\lambda} = \frac{(n-r)!}{n!} \oint \dots \oint M_{\mu; r_1, \dots, r_k}(z_1, \dots, z_k) dz_1 \dots dz_k; \quad (41)$$

$$M_{\mu; \mathbf{r}}(\mathbf{z}) = \left(\prod_{i=1}^k M_{\mu; r_i}(z_i) \right) \times \prod_{1 \leq i < j \leq k} \left(\frac{(z_i - z_j)(z_i - z_j - r_i + r_j)}{(z_i - z_j - r_i)(z_i - z_j + r_j)} \right).$$

Here an admissible configuration of the contours has the contour \mathcal{C}_1 of z_1 encompassing μ_1, \dots, μ_n , the contour \mathcal{C}_2 of z_2 encompassing $\mathcal{C}_1, \mathcal{C}_1 \pm r_1, \mathcal{C}_1 \pm r_2$, the contour \mathcal{C}_3 of z_3 encompassing $\mathcal{C}_2, \mathcal{C}_2 \pm r_2, \mathcal{C}_2 \pm r_3$, etc.

When $\rho = (r, 1^{n-r})$ ([13], p. 118) gives

$$\frac{\chi_\rho^\lambda}{f^\lambda} = \frac{n!}{(n-r)!} \sum_{i=1}^n \frac{\mu_i!}{(\mu_i - r)!} \prod_{j \neq i} \frac{\mu_i - \mu_j - r}{\mu_i - \mu_j}. \quad (42)$$

This is visibly equal to the sum of the residues of $M_{\mu, r}(z)$, which shows (40).

In the general case, ([13], p.117, 6 and 7) gives that χ_ρ^λ is the coefficient on \mathbf{x}^μ in

$$(\sum x_i^{r_1})(\sum x_i^{r_2}) \dots (\sum x_i^{r_k})(\sum x_i)^{n-r} \Delta(\mathbf{x})$$

where $\Delta(\mathbf{x}) = \Delta(x_1, \dots, x_n)$ is the Vandermonde. In particular, he deduces the dimension formula

$$M_\lambda = \chi_{1^n}^\lambda = \frac{n!}{\mu!} \Delta(\mu)$$

and in the case $\rho = (r, 1^{n-r})$ of a single cycle he gives a formula for χ_ρ^λ , from which he deduces (42). Write $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ with the 1 in the i th slot. In the case $\rho = (r, 1^{n-r})$ of a single cycle, Macdonald's formula reads

$$\chi_\rho^\lambda = (n-r)! \sum_{i=1}^n \frac{\Delta(\mu - r\mathbf{e}_i)}{(\mu - r\mathbf{e}_i)!}.$$

For general $\rho = (r_1, \dots, r_k, 1^{n-r})$ the analogous formula is

$$\chi_\rho^\lambda = (n-r)! \sum_{1 \leq i_1, \dots, i_k \leq n} \frac{\Delta(\mu - r_1\mathbf{e}_{i_1} - \dots - r_k\mathbf{e}_{i_k})}{(\mu - r_1\mathbf{e}_{i_1} - \dots - r_k\mathbf{e}_{i_k})!}.$$

Thus the character ratio is given by

$$\frac{\chi_\rho^\lambda}{f^\lambda} = \frac{(n-r)!}{n!} \sum_{1 \leq i_1, \dots, i_k \leq n} \frac{\frac{\mu!}{(\mu - r\mathbf{e}_i)!} \Delta(\mu - r_1\mathbf{e}_{i_1} - \dots - r_k\mathbf{e}_{i_k})}{(\mu - r_1\mathbf{e}_{i_1} - \dots - r_k\mathbf{e}_{i_k})! \Delta(\mu)}. \quad (43)$$

We prove the equality of (43) with (41) by induction. First notice that if we interpret $\frac{\mu!}{(\mu - r\mathbf{e}_i)!}$ as μ_i^r in (42), then the equality between (40) and (42) actually holds for μ_i that are arbitrary pairwise distinct complex numbers. We include this more general condition in the inductive assumption.

Suppose that the case $k - 1$ is settled and we wish to prove case k . We assume that an acceptable configuration of k contours has been fixed. Invoking the inductive assumption we may write (43) as $[\check{\mathbf{r}}_1$ indicates the string r_2, \dots, r_k with r_1 omitted, and similarly $\check{\mathbf{z}}_1]$

$$\begin{aligned} & \frac{n!}{(n-r)!} \sum_{1 \leq i_1 \leq n} \mu_{i_1}^{r_1} \frac{\Delta(\mu - r_1 e_{i_1})}{\Delta(\mu)} \sum_{1 \leq i_2, \dots, i_k \leq n} \frac{(\mu - r_1 e_{i_1})!}{(\mu - \sum_{j=1}^k r_j e_{i_j})!} \frac{\Delta(\mu - \sum_{j=1}^k r_j e_{i_j})}{\Delta(\mu - r_1 e_{i_1})} \\ &= \frac{n!}{(n-r)!} \oint \dots \oint \sum_{1 \leq i_1 \leq n} \left[\frac{r_1}{z_1} \prod_{j \neq i_1} \frac{z_1 - \mu_j - r_1}{z_1 - \mu_j} M_{\mu - r_1 e_{i_1}; \check{\mathbf{r}}_1}(\check{\mathbf{z}}_1) \right]_{z_1 = \mu_{i_1}} dz_2 \dots dz_k. \end{aligned} \quad (44)$$

Since

$$M_{\mu - r_1 e_{i_1}; \check{\mathbf{r}}_1}(\check{\mathbf{z}}_1) = M_{\mu; \check{\mathbf{r}}_1}(\check{\mathbf{z}}_1) \prod_{2 \leq j \leq k} \left(\frac{(z_1 - z_j)(z_1 - z_j - r_1 + r_j)}{(z_1 - z_j - r_1)(z_1 - z_j + r_j)} \right) \Big|_{z_1 = \mu_{i_1}}$$

and

$$\frac{r_1}{z_1} \prod_{j \neq i_1} \frac{z_1 - \mu_j - r_1}{z_1 - \mu_j} \Big|_{z_1 = \mu_{i_1}}, \quad i_1 = 1, 2, \dots, n$$

are the residues of $M_\mu(z_1)$, we see that the right side of (44) is equal to (41) once the residues in the inner integral are evaluated.

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